Variational dynamics of ultracold fermion-boson mixtures

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Abstract

We study the mean-field dynamic of a interacting boson-fermion mixture in an harmonic trap, by means of a Gaussian variational ansatz. We find concentric stationary solutions in the repulsive regime and in the regime of weak boson-fermion attraction. Above a threshold value of the repulsion parameter, a stable symmetry-broken stationary solution bifurcates, characterized by the displacement of the center of mass of the bosonic and fermionic clouds in opposite directions away from the trap center. We observe linear and nonlinear oscillations of the width and center of mass of the boson and fermion distributions whenever we let the system evolve starting from a non stationary initial condition.

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1 Introduction

Ultracold atoms in electromagnetic traps provide a fashionable playground for exploring many-body dynamics in simple idealized configurations or in complicated interesting situations [1, 2, 3]. In particular rarefied boson-fermion mixtures [4, 5] present a rich bulk phase diagram [6], and nontrivial dynamical properties where experimental investigation is only starting [3, 5, 7, 8, 9, 10]. In the present thesis, we consider the rather idealized case of a zero-temperature dilute mixture composed of \( N_b \) atomic bosons forming an interesting Bose-Einstein condensate and \( N_f \) attractive atomic fermions in the superfluid state. The mixture is trapped by an external optical field which induces a spherically-symmetric harmonic confinement.

At extremely low temperatures dilute bosonic atoms form a pure Bose-Einstein condensate, which is a bosonic superfluid. Also attractive fermions equally distributed into two hyperfine states form a superfluid of Cooper pairs: a fermionic superfluid. Both bosonic and fermionic superfluids are characterized by an irrotational velocity field. We use the quantum irrotational hydrodynamics (QIH) to study our superfluid Bose-Fermi mixture.

We assume that the motion of the boson and fermion densities is driven by two coupled fields: the macroscopic wave function \( \psi_b(r, z, t) \) of the Bose-Einstein condensate and the Ginzburg-Landau order parameter \( \psi_f(r, z, t) \) of the Cooper pairs of fermionic atoms.

We use for the fields \( \psi_j(r, z, t) \) \((j = b, f)\) a cylindrically-symmetric Gaussian ansatz. This ansatz depends on variational parameters which characterize the position and width of each atomic clouds. Within this simplifying hypothesis we integrate the action over spatial variables and obtain an effective action which provides the equations of motion for the variational parameters. We study the stationary solutions and the numerical dynamics of the parameters.

2 The formalism

The QIH action functional \( A \) of the Bose-Fermi mixture is given by

\[
A = \int \left( \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{bf} \right) d^3r \, dt,
\] (1)
where $\mathcal{L}_b$ is the bosonic Lagrangian density, $\mathcal{L}_f$ is the fermionic one, and $\mathcal{L}_{bf}$ is the Lagrangian density of the Bose-Fermi interaction.

We consider the bosonic Lagrangian density $\mathcal{L}_b$ defined by

$$\mathcal{L}_b = \frac{i}{2} \hbar (\psi_b^* \frac{\partial}{\partial t} \psi_b - \psi_b \frac{\partial}{\partial t} \psi_b^*) + \frac{\hbar^2}{2m_b} \psi_b^* \nabla^2 \psi_b - \frac{1}{2} g_b |\psi_b|^4 - V_b |\psi_b|^2,$$

where $V_b(r)$ is the external potential acting on the atomic bosons. $g_b$ is the boson-boson interaction strength associated to the s-wave interatomic scattering length $a_b$ by the formula $g_b = \frac{4\pi \hbar^2 a_b}{m_b}$ with $m_b$ the mass of the single bosonic atom. $\psi_b(r,t)$ is the macroscopic wave function of the Bose-Einstein condensate, such that $n_b(r,t) = |\psi_b(r,t)|^2$ is the local number density of bosonic atoms.

The total number of bosonic atoms reads

$$N_b = \int |\psi_b(r,t)|^2 d^3r,$$

where integration extends through all three dimensional space.

The fermionic Lagrangian density $\mathcal{L}_f$ is defined as

$$\mathcal{L}_f = \frac{i}{2} \hbar (\psi_f^* \frac{\partial}{\partial t} \psi_f - \psi_f \frac{\partial}{\partial t} \psi_f^*) + \frac{\hbar^2}{4m_f} \psi_f^* \nabla^2 \psi_f - 2\varepsilon (2|\psi_f|^2)|\psi_f|^2 - 2V_f |\psi_f|^2,$$

where $2V_f(r)$ is the external potential acting on a Cooper pair of mass $2m_f$, where $m_f$ is the mass of a single fermionic atom. $\varepsilon$ is the bulk energy per particle of the fermions, which is a function of the number density $n_f(r,t) = 2|\psi_f(r,t)|^2$ where $\psi_f(r,t)$ is the Ginzburg-Landau order parameter of the superfluid Fermi gas, i.e. the macroscopic wave function of Cooper pairs.

The total number of fermionic atoms is given by the normalization

$$N_f = 2 \int |\psi_f(r,t)|^2 d^3r.$$

The bulk energy per particle of a two-spin component attractive Fermi gas can be expressed by the following expression

$$\varepsilon(n_f) = \frac{3 \hbar^2 k_f^2}{5 2m_f} G \left( \frac{1}{a_h k_F} \right),$$

where $G(y)$ depends on the local inverse interaction parameter $y = 1/(a_h k_F(n_f))$ is a function of position $r$ and time $t$, because the Fermi momentum is $k_f(r,t) = (3\pi^2 n_f(r,t))^{1/3}$ the local Fermi wave vector. $a_f$ is the fermion-fermion scattering length.
In the experiments on the BCS-BEC crossover the scattering length $a_f$ of superfluid fermions is changed by using an external magnetic field: the Feshbach resonance technique. $a_f$ is varied from large negative values, so that $y \ll -1$ and the Fermi superfluid is a BCS state of weakly bound Cooper pairs, to large positive values, so that $y \gg 1$ and the fermions form a weakly repulsive Bose gas of molecules. In the first case we have $G(y) = 1 + 10/(9\pi y) + O(1/y^2)$, in the second case $G(y) = 5a_M/(18\pi a_M y) + O(y^{5/2})$, giving the asymptotic behavior of $G(y)$. Another special limit is $y = 0$, the so called unitarity limit, in which one expects that the energy per particle is proportional to a non-interacting Fermi gas with coefficient $G(0) = 0.42$.

An analytical formula of the universal function $G(y)$ which fits the numerical results from Monte Carlo simulations [11] and obeys the above asymptotic expressions is $G(y) = \alpha_1 - \alpha_2 \arctan \left( \frac{\beta_1 + |y|}{\beta_2 + |y|} \right)$, with the specific value for $\alpha_i$ and $\beta_i$ given in Ref.[12].

The Lagrangian density $\mathcal{L}_{bf}$ of the Bose-Fermi interaction is given by the simple form

$$\mathcal{L}_{bf} = -2g_{bf} |\psi_b|^2 |\psi_f|^2,$$

where $g_{bf} = \frac{2\pi \hbar^2 a_{bf}}{m_r}$ where is the Bose-Fermi interaction strength, with $m_r = m_b m_f / (m_b + m_f)$ as the reduced mass and $a_{bf}$ the Boson-Fermion scattering length.

Finally the external trapping potential is taken of the simple harmonic form,

$$V_j(x, y, z) = \frac{1}{2} m_j \omega_j^2 (x^2 + y^2 + z^2), \quad j = b, f.$$

We assume generally different trapping frequencies for bosons and fermions. In the present preliminary investigation we assume spherical symmetry of the trap. The trapping potential for bosons defines a natural harmonic-oscillator length scale $a_h = h^{1/2} m_b^{-1/2} \omega_b^{-1/2}$.

### 2.1 Dimensionless units

The Lagrangian is conveniently expressed in dimensionless form by using natural units for each physical quantity, as listed in Table 1. Tildes identify the quantities in reduced dimensionless units, e.g. $\tilde{t} = \omega_b t, \tilde{r} = r/a_h, \tilde{L}_j = L_j/\hbar \omega_b, \tilde{\psi}_j = a_h^{3/2} \psi_j$. 
Physical quantity | units used
--- | ---
Mass | $m_b$
Time | $\omega_b^{-1}$
Length | $a_h = \sqrt{\frac{\hbar}{m_b \omega_b}}$
Energy | $\hbar \omega_b$
Density | $a_h^3$

Table 1: Natural units used in the calculations

The bosonic part is

$$\tilde{\mathcal{L}}_b = \frac{a_b^3}{\hbar \omega_b} \mathcal{L}_b = \frac{a_b^3}{\hbar \omega_b} \left[ \frac{i \hbar a_b^{-3}}{2} (\tilde{\psi}_b^* \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_b - \tilde{\psi}_b \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_b^*) + \frac{\hbar^2}{2 m_b a_b^3} \tilde{\psi}_b^* \nabla^2 \tilde{\psi}_b + \frac{1}{2} \frac{1}{\hbar \omega_b a_b^3} \frac{4 \pi \hbar^2 a_b^3}{m_b} |\tilde{\psi}_b|^4 - a_b^{-1} m_b \omega_b^2 \frac{1}{2} \tilde{r}^2 |\tilde{\psi}_b|^2 \right] =$$

$$= \frac{i}{2} (\tilde{\psi}_b^* \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_b - \tilde{\psi}_b \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_b^*) + \frac{1}{2} \tilde{\psi}_b^* \nabla^2 \tilde{\psi}_b - \tilde{g}_b \frac{1}{2} |\tilde{\psi}_b|^4 - \frac{1}{2} \tilde{r}^2 |\tilde{\psi}_b|^2. \quad (7)$$

Likewise the fermionic part is

$$\tilde{\mathcal{L}}_f = \frac{a_f^3}{\hbar \omega_f} \mathcal{L}_f = \frac{i}{2} (\tilde{\psi}_f^* \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_f - \tilde{\psi}_f \frac{\partial}{\partial \tilde{\tau}} \tilde{\psi}_f^*) + \lambda \frac{1}{4} \tilde{\psi}_f^* \nabla^2 \tilde{\psi}_f - 2 \tilde{\varepsilon} (2|\tilde{\psi}_f|^2) |\tilde{\psi}_f|^2 - \frac{1}{4 \lambda} \tilde{r}^2 |\tilde{\psi}_f|^2 \quad (8)$$

where $\tilde{\varepsilon} (2|\tilde{\psi}_f|^2) = \frac{1}{\hbar \omega_f} \varepsilon (2|\tilde{\psi}_f|^2)$.

Finally, the interaction Lagrangian term is

$$\tilde{\mathcal{L}}_{bf} = \frac{a_b^3}{\hbar \omega_b} \mathcal{L}_{bf} = -2 \tilde{g}_b |\tilde{\psi}_b|^2 |\tilde{\psi}_f|^2. \quad (9)$$

The several dimensionless parameters introduced above are:

$$\lambda = \frac{m_b}{m_f}, \quad \lambda' = \frac{m_b \omega_b}{m_f \omega_f}, \quad \tilde{g}_b = \frac{4 \pi a_b}{a_b^3}, \quad \tilde{g}_f = \frac{2 \pi a_b m_b}{a_h m_r}. \quad (10)$$

Hence, we shall use the rescaled units and suppress all tildes from now on, to keep the notation simple.

### 3 The Gaussian ansatz

Even though the external potential is assumed to be spherically-symmetric, we choose a cylindrical symmetric Gaussian ansatz allowing for independent axial
displacements of the Fermion and Boson clouds
\[ \psi_j(x, y, z, t) = \psi_j(r, z, t) = A_j e^{-\frac{r^2 + (z - z_0j(t))^2}{2\sigma_j^2}} e^{i[\alpha_j(t)(r^2 + z^2) + p_j(t)(z - z_0j(t))]} . \] (11)

The 4 + 4 parameters defining the Gaussian fields carry a time dependence which is often left implicit
\[ \sigma_j = \sigma_j(t), \quad \alpha_j = \alpha_j(t), \quad p_j = p_j(t), \quad z_0j = z_0j(t), \] with \( j = b, f \). The field normalization is
\[ A_b = \sqrt{\frac{N_b}{\sigma^3_b\pi^{3/2}}}, \quad A_f = \sqrt{\frac{N_f}{2\sigma^3_f\pi^{3/2}}} . \] (12)

We substitute the ansatz into Eqs.(7), (8) and (9) and integrate the total Lagrangian density over space:
\[ L_{TOT} = \int (L_b + L_f + L_{bf}) d^3r = L_b + L_f + L_{bf} . \] (13)

The integration is carried out in cylindrical coordinates.

The resulting bosonic part of Lagrangian is given by:
\[ \frac{L_b}{N_b} = -\frac{3}{2} \dot{\alpha}_b\sigma_b^2 + N_b\dot{z}_0b - \dot{\alpha}_b z_0^2b \\
- \left[ \frac{3}{4} \frac{1}{\sigma_b^2} + 3\alpha_b^2\sigma_b^2 + 2\alpha_b^2z_0^2b + \frac{1}{2}p_b^2 + 2p_b\dot{\alpha}_b z_0b \right] \\
- \left[ \frac{g_b}{2} \frac{N_b}{\sigma_b^3(2\pi)^{3/2}} + \frac{3}{4}\frac{1}{\sigma_b^2} + \frac{1}{2}\frac{z_0^2b}{2} \right] . \] (14)

The terms in the first line are generated by time derivatives in Eq.(7). The first term is the same that one finds in the spherical case, the second and the third term arise from the motion of the Gaussian center of mass. In the second line we have the terms generated from the Laplacian. The first and the second term are ordinary terms of the spherical case, the three other terms again relate to center of cent-re-mass motion. The third line contains the self-interaction term and the terms derived from the external harmonic potential.

The fermionic part of the Lagrangian is given by
\[ \frac{L_f}{N_f} = -\frac{3}{4} \dot{\alpha}_f\sigma_f^2 + \frac{1}{2}p_f\dot{z}_0f - \frac{1}{2}\dot{\alpha}_f z_0^2f \\
- \left[ \frac{3}{16} \frac{1}{\sigma_f^2} + \frac{3}{4}\alpha_f^2\sigma_f^2 + \frac{1}{8}p_f^2 + \frac{1}{2}\alpha_f^2z_0^2f + \frac{1}{2}p_f\dot{\alpha}_f z_0f \right] \lambda \\
- \left[ \frac{4}{\sqrt{\pi}} \Phi \left( \frac{N_f}{\sigma^3_f\pi^{3/2}} \right) + \left( \frac{3}{4}\sigma_f^2 + \frac{1}{2}\frac{z_0^2f}{\lambda} \right) \right] , \]
with

$$\Phi(x) = \int_{0}^{+\infty} \varepsilon(xe^{-\xi^2})e^{-\xi^2}d\xi.$$  \hspace{1cm} (16)

The Bose-Fermi interaction takes the following form:

$$L_{bf} = -g_{bf}N_fN_b \left[ \frac{1}{\pi (\sigma_b^2 + \sigma_f^2)} \right]^\frac{3}{2} e^{-\frac{(z_{0f} - z_{0b})^2}{(\sigma_f^2 + \sigma_b^2)}}.$$ \hspace{1cm} (17)

It is convenient to organize the terms in $L_{TOT}$ into two groups, $L_{TOT} = T - E$, as follows

$$T = - \frac{3}{2} N_b[\sigma_b^2\dot{\alpha}_b + 2\alpha_b^2\sigma_b^2] - N_b[\dot{\alpha}_b z_{0b}^2 - p_b \dot{z}_b] - 2 N_b[\alpha_b^2 z_{0b}^2 + p_b \alpha_b z_{0b}] \hspace{1cm} (18)$$

$$- \frac{3}{4} N_f[\sigma_f^2\dot{\alpha}_f + \lambda \alpha_f^2\sigma_f^2] - \frac{N_f}{2}[\alpha_f z_{0f}^2 - p_f \dot{z}_f]$$

$$- \frac{N_f}{2}[\alpha_f^2 z_{0f}^2 \lambda + p_f \alpha_f z_{0f} \lambda],$$

$$E = \frac{3}{4} N_b[\frac{1}{\sigma_b^2} + \sigma_b^2] + \frac{3}{4} N_f[\frac{\lambda}{4\sigma_f^2} + \frac{\sigma_f^2}{\lambda}] + \frac{g_b N_b^2}{2 (2\pi)^{\frac{3}{2}}} \frac{1}{\sigma_b^2} +$$

$$+ \frac{N_f}{2}[\frac{z_{0f}^2}{\lambda} + \frac{1}{4\sigma_f^2} \lambda] + \frac{N_b}{2}[p_b^2 + z_{0b}^2] +$$

$$+ \frac{4 N_f}{\sqrt{\pi}} \Phi\left( \frac{N_f}{\sigma_f^2 \pi^\frac{3}{2}} \right) + \frac{g_{bf} N_f N_b}{\pi^\frac{3}{2} (\sigma_f^2 + \sigma_b^2)^\frac{3}{2}} e^{-\frac{(z_{0f} - z_{0b})^2}{(\sigma_f^2 + \sigma_b^2)}}. \hspace{1cm} (19)$$

### 4 Euler-Lagrange equation

We write now the Euler-Lagrange equation for the boson parameters. To simplify the notation we use the shorthand $S^2 = \sigma_f^2 + \sigma_b^2$, $Z = (z_{0f} - z_{0b})$.

From the $\alpha_b$-Euler-Lagrange equation $\frac{\partial L}{\partial \dot{\alpha}_b} - \frac{\partial L}{\partial \alpha_b} = 0$ we obtain:

$$\dot{\sigma}_b = -\frac{2}{3} z_{0b} \dot{z}_{0b} + 2 \alpha_b \sigma_b + \frac{4}{3} \alpha_b z_{0b}^2 + 2 \frac{p_b z_{0b}}{\sigma_b}.$$ \hspace{1cm} (20)

From the equation $\frac{\partial L}{\partial \dot{p}_b} - \frac{\partial L}{\partial p_b} = 0$ we obtain:

$$\dot{z}_{0b} = p_b + 2 \alpha_b z_{0b}.$$ \hspace{1cm} (21)
We use both equations, by inserting Eqs.(21) into (20) and obtain
\[ \alpha_b = \frac{\dot{\sigma}_b}{2\sigma_b}. \]  
(22)

We use now \[ \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{z}_0} - \frac{\partial L}{\partial z_0} = 0, \] i.e.
\[ \dot{p}_b = -4\alpha_b^2 z_0 - 2p_b \alpha_b - z_0 \frac{\dot{\alpha}_b}{\sigma_b} - g_{bf} N_f \pi^{-\frac{3}{2}} e^{-\left(\frac{z_0}{\sigma_b}\right)^2} \frac{2Z}{S^5}. \]  
(23)

\[ \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\sigma}_b} - \frac{\partial L}{\partial \sigma_b} = 0 \] yields the last equation of motion for the boson parameters
\[ \dot{\sigma}_b = \frac{1}{2\sigma_b^2} - 2\alpha_b^2 - \frac{1}{2} \frac{\sigma_b}{(2\pi)^\frac{3}{2} \sigma_b^5} \left[ -e^{-\left(\frac{z_0}{\sigma_b}\right)^2} + \frac{2Z^2 e^{-\left(\frac{z_0}{\sigma_b}\right)^2}}{3S^7} \right]. \]  
(24)

We plug now equations (24), (21) and (22) into equation (23) to obtain:
\[ \dot{p}_b = \frac{z_{0b}}{\sigma_b^2} + \frac{\dot{\sigma}_b^2}{\sigma_b^2} z_{0b} - z_{0b} \frac{\dot{\sigma}_b}{\sigma_b} - \frac{\tilde{g}_{bf} N_f}{(2\pi)^\frac{3}{2}} \frac{1}{\sigma_b^5} - \frac{\tilde{g}_{bf} N_f}{(2\pi)^\frac{3}{2} \sigma_b^5} \frac{1}{\sigma_b^5} \left[ -e^{-\left(\frac{z_0}{\sigma_b}\right)^2} + \frac{2Z^2 e^{-\left(\frac{z_0}{\sigma_b}\right)^2}}{3S^7} \right]. \]  
(25)

We substitute \( \dot{\alpha}_b \) from Eq. (24) and \( p_b \) from Eq. (21) into Eq. (23) and then we use the relation for \( \alpha_b \) in (22) to obtain an equation for \( \dot{p}_b \) involving only \( \sigma_b \) and \( z_0 \) for bosons and fermions at the right-hand side.

The Euler-Lagrange equations for fermion parameters are:
\[ \dot{\sigma}_f = \frac{1}{2} \alpha_f \sigma_f \lambda, \]  
(26)
\[ \dot{z}_0f = \frac{1}{2} \rho_f \lambda + \alpha_f z_0f \lambda, \]  
(27)
\[ \dot{\alpha}_f = -\lambda \alpha_f^2 - \frac{1}{\lambda} \frac{\lambda}{4\sigma_f^4} + \frac{8N_f}{\pi^2 \sigma_f^6} \Phi' \left( \frac{N_f}{\sigma_f^2 \pi^2} \right) + \frac{2g_{bf} N_f e^{-\left(\frac{z_0}{\sigma_b}\right)^2} \pi^{-\frac{3}{2}}}{S^{-5} - \frac{2Z^2 S^{-7}}{3}} \right]. \]  
(28)
Like for bosons we obtain an expression for \( \dot{p}_f \):

\[
\dot{p}_f = -2\alpha_f z_{0f} \lambda - p_f \alpha_f \lambda - \frac{2z_{0f}}{\lambda'} - 2\alpha_f z_{0f} + 2g_{bf} N_f \pi^{-\frac{3}{2}} e^{-\frac{(\xi_s)^{2}}{Z/S^5}} = (29)
\]

\[
= \frac{8\sigma_f^2 z_{0f}}{\lambda \sigma_f^2} - \frac{z_{0f} \lambda}{2\sigma_f^4} - \frac{4z_{0f} \sigma_f}{\lambda \sigma_f^2} - \frac{16N_f z_{0f}}{\pi^2 \sigma_f^4} \Phi'(\frac{N_f}{\sigma_f^2 \pi^2}) + \nonumber + 2g_{bf} N_b e^{-\frac{(\xi_s)^{2}}{Z/S^5}} 2\pi^{-\frac{3}{2}} \left[ \frac{Z}{S^5} - \frac{z_{0f}}{S^5} + \frac{2z_{0f} Z^2}{3S^7} \right].
\]

Differences between fermion and boson include the \( \lambda \) factor, the self-interaction term, and the sign of one term in \( \dot{p}_f \). Several factors of 2 come from \( \psi_f \) describing the motion of Cooper pairs, as opposed to individual bosons.

The terms involving \( \Phi' \) in Eqs.(28) and (29) derive from: \( \frac{\partial}{\partial \sigma_f} \Phi = \frac{\partial x}{\partial \sigma} \frac{\partial x}{\partial \Phi} \Phi(x) = -\frac{3N_f}{\sigma_f^2} \Phi'(x) \), where \( x = \frac{N_f}{\sigma_f^2 \pi^2} \). The integral definition of \( \Phi \) yields

\[
\Phi'(x) = \frac{\partial}{\partial x} \int_0^{+\infty} \xi^2 \varepsilon(xe^{-\xi^2})d\xi = \int_0^{+\infty} \varepsilon(xe^{-\xi^2})\xi^2 e^{-2\xi^2}. \tag{30}
\]

We substitute the relation \( \frac{\partial}{\partial x} \left[ \varepsilon(xe^{-\xi^2}) \right] = \varepsilon(xe^{-\xi^2})xe^{-\xi^2}(-2\xi) \) into the integral (30) and integrate by parts, to obtain:

\[
\int_0^{+\infty} \varepsilon(xe^{-\xi^2})xe^{-\xi^2} \left[ -\frac{1}{2x^2} \right] e^{-\xi^2}d\xi = \tag{31}
\]

\[
= \varepsilon(xe^{-\xi^2}) \left[ \frac{\xi^2 e^{-\xi^2}}{2x} \right] \bigg|_0^{+\infty} - \int_0^{+\infty} \varepsilon(xe^{-\xi^2}) \left[ -\frac{e^{-\xi^2}}{2x} + \frac{\xi^2 e^{-\xi^2}}{x} \right] d\xi = \nonumber \]

\[
= \frac{1}{x} \int_0^{+\infty} \varepsilon(xe^{-\xi^2}) e^{-\xi^2} \left[ \frac{1}{2} - \xi^2 \right] d\xi = \Phi'(x)
\]

The last equation is the expression we use to evaluate \( \Phi' \) numerically.

### 4.1 Stationary points

We first study the eight equations of motion for the eight parameters derived above, to find the stationary points. These equilibrium positions are characterized by all parameters being independent of \( t \), so that all that terms vanish. \( \sigma_j = z_{0j} = \)
\( \dot{\alpha}_j = \dot{\rho}_j = 0. \ (j = b, f) \) we obtain a set of algebraic equations:

\[
\frac{1}{2 \sigma_b^2} \sigma_b - \frac{1}{2} \cdot \frac{1}{2} g_b \frac{N_b}{(2\pi)^2} \frac{1}{\sigma_b^2} - g_b f N_f \pi^{-\frac{3}{2}} \left[ -e^{-\left(\frac{Z}{S}\right)^2} + \frac{2}{3} Z^2 e^{-\left(\frac{Z}{S}\right)^2} \right] = 0,
\]

(32)

\[
\frac{z_{0b} - g_b f N_f \pi^{-\frac{3}{2}} e^{-\left(\frac{Z}{S}\right)^2} \frac{2Z}{S^5}}{S^5} = 0,
\]

(33)

\[
\frac{z_{0f} - g_b f N_b \pi^{-\frac{3}{2}} e^{-\left(\frac{Z}{S}\right)^2} \frac{2Z}{S^5}}{S^5} = 0,
\]

(34)

where we have used the simpler relations \( \alpha_j = \rho_j = 0, \) with \( j = b, f, \) derived from Eqs. (21), (22), (26), (27).

As an example, we consider a mixture of \( ^6\text{Li} \) and \( ^{23}\text{Na} \) with \( N_b = 2 \cdot 10^9, \) \( N_f = 6 \cdot 10^5, \) with the parameters indicated in Fig.1, and explore the role of the parameter \( g_b f \) gauging the boson-fermion interaction. Through a numerical calculation we obtain that in the attractive regime \( g_b f < 0 \) the boson and fermion clouds reach the equilibrium at \( z_{0j} = 0, \) as one expects. Instead, with \( g_b f > 0, \) we observe that at first the equilibrium positions remain around zero, but when \( g_b f \) increases above a critical value, here \( g_b f = 0.0048, \) the center of both fermion and boson clouds start to leave the origin. As one expects, the bosons displacement \( z_{0b} \) assumes much smaller value than that of fermions at the same \( g_b f, \) due to the much larger number of bosons that keeps their position much more fixed in interaction with the external harmonic potential. The typical behavior is shown in Fig.1.

The same calculation provides the stationary value of the two \( \sigma_j. \) The results are plotted in Fig.2. We observe that for \( g_b f < 0.0048 \) the decreasing attraction and then the increasing repulsive regime causes an increasing spread of the widths, thus an increase of both \( \sigma_b \) and \( \sigma_f. \) This increasing regime continues along the unstable solution, characterized by \( z_{0f} = z_{0b} = 0 \) (dotted lines in Fig.2), but at \( g_b f \geq 0.00485 \) a new displaced stable solution arises with \( z_{0j} \neq 0, \) see Fig.1. The two clouds separate, and for further increase of \( g_b f \) the values of \( \sigma_j \) decrease. Note that the variations of \( \sigma_f \) are much larger than those of \( \sigma_b, \) again due to \( N_f \ll N_b. \)

### 4.2 Time evolution

We investigate now the time evolution of the boson-fermion mixture through the numerical solution of the Euler-Lagrange equations (21), (22), (24), (25), (26),
Figure 1: The stationary values of \( z_{0f} \) (solid line) and \( z_{0b} \) (dot dashed line), as a function of the dimensionless boson-fermion interaction strength \( g_{bf} \) for a mixture of \( N_b = 2 \cdot 10^9 \) \( ^{23}\text{Na} \) and \( N_f = 6 \cdot 10^5 \) \( ^6\text{Li} \). We consider \( a_b = 10 \) nm, \( a_f = 10 \) nm, \( \omega_b = 200\pi \) Hz, and \( \omega_f = 321.81\pi \) Hz, so that \( \lambda = 3.833 \), \( \lambda' = 2.393 \), \( g_b = 0.0473636 \), and \( g_f = 0.00376908 \). The arrow indicates the critical value of \( g_{bf} \) beyond which nonzero equilibrium positions arise.
Figure 2: The stationary values of $\sigma_f$ (solid line) and $\sigma_b$ (dot dashed line), as a function of $g_{bf}$ for a mixture of $N_b = 2 \cdot 10^9$ $^{23}$Na and $N_f = 6 \cdot 10^5$ $^6$Li. The unstable solution corresponding to $z_{0f} = z_{0b} = 0$ is depicted by dotted lines. This calculation is based on the same parameters as in Fig.1. The peak of boson and fermion widths coincide with the critical value $g_{bf} = 0.048$, where a nonzero displacement of the boson and fermion Gaussian starts off.
Figure 3: Time calculation in the weakly repulsive regime. \( N_b = 2 \cdot 10^9 \) \(^{23}\)Na, \( N_f = 6 \cdot 10^5 \) \(^6\)Li atoms. We assume \( a_b = 10 \) nm, \( a_f = 10 \) nm, \( a_{bf} = 10 \) nm, \( \omega_b = 200\pi \) Hz, \( \omega_f = 321.81\pi \) Hz, \( \lambda = \frac{m_b}{m_f} = 3.833 \), \( \lambda' = \frac{m_a}{m_f} \omega_f = 2.393 \), \( a_h = \left( \frac{\hbar}{m_a \omega_b} \right)^2 = 2.65317 \cdot 10^{-6} \) m, \( m_r = \frac{m_b m_f}{m_b + m_f} = 7.90188 \cdot 10^{-27} \) Kg, \( g_b = \frac{4\pi m_b}{\alpha_b \alpha_b} = 0.0473636 \), \( g_f = \frac{a_f}{\alpha_b} = 0.00376908 \) and \( g_{bf} = \frac{2\pi a_h}{a_h m_r} = 0.004 \). The initial conditions are: \( z_{0b}(0) = 0 \) (equilibrium), \( \sigma_{0b}(0) = 22.6909 \) (equilibrium), \( \sigma_f(0) = 6.89197 \) (equilibrium), and \( \alpha_j(0) = p_j(0) = 0 \).

We first investigate a weakly repulsive regime to evaluate the relations between the two \( z_{0j} \) and the two \( \sigma_j \). We start off the two \( \sigma_j \) at their stationary

(27), (28), (29).
Figure 4: Time evolution in the strongly-repulsive regime. For the parameters see Fig.3, except $g_{bf} = 0.8$. The initial conditions are: $z_{0b}(0) = -2$, $z_{0f}(0) = 50$, $\sigma_{b}(0) = 22.70$, $\sigma_{f}(0) = 3.3$. The equilibrium position: $z_{0b}(eq) = -0.00666513$, $z_{0f}(eq) = 53.1683$, $\sigma_{b}(eq) = 22.6889$, $\sigma_{f}(eq) = 3.26125$. Note that the initial $z_{0b}$ is fixed a value very far from the equilibrium point.

value not to disturb the other variables too strongly. In Fig.3 we see that with the only displacement of $z_{0f}$ from the stationary point we have the excitation also of a visible oscillation of of $z_{0b}$. This oscillatory motion is disturbed by the exchange of energy with the two widths $\sigma_j$: even if their oscillations are small this collective interaction between the variables causes the modulation of the peaks in $z_{0b}$. An antiphase correlation between $z_{0b}$ and $z_{0f}$ is recognizable.

The second regime we study is the strong repulsion regime. The interaction term is much above the critical point, so we expect that the oscillations of the variables being modulated by this large repulsion value. We set the two $\sigma_j$ in a value near the equilibrium value to observe their weak interactions with the other variables. The strong interaction between the two symmetry-broken $z_{0j}$ tends to make the clouds push away each other, see Fig.4. Instead, contrary to what we observed for weak repulsion, the fermions oscillate at the same basic frequency.
Figure 5: Attractive regime: the parameters are the same in Fig.3, but $g_{bf} = -0.1$. The initial conditions: $z_{0b}(0) = -0.001$, $z_{0f}(0) = 0.001$, $\sigma_b(0) = 21$, $\sigma_f(0) = 2$. The equilibrium position are: $z_{0b}(eq) = 0$, $z_{0f}(eq) = 0$, $\sigma_b(eq) = 22.6744$, $\sigma_f(eq) = 2.157383$. Initially, both $z_{0j}$ differ from the stationary point by a small value.

and with the same phase as the bosons. We can deduce that in this regime the oscillations of $z_{0b}$ decides the dynamic of the system even though the fermions have an initial $z_{0f}$ different from the equilibrium point. The influence of the initial value of $z_{0f}$ remain confined in the higher-frequency oscillations of $z_{0f}$. $\sigma_b$ shows a fairly harmonic oscillation. $\sigma_f$ shows an evident exchange of energy with $z_{0f}$. The large oscillations of the fermionic parameters are due to the relatively small number of fermions compared to bosons.

Finally we investigate the attractive regime, $z_{0j}$ reach equilibrium at zero. We study the dynamics with $z_{0j}(0) \neq 0$ which shows the attractive interaction that one clouds exerts on the other. In Fig.5 we see clearly that the $z_{0b}$ attract in a strong manner $z_{0f}$ that follows the oscillating dynamics of $z_{0b}$ and also makes faster oscillations on its own. The dynamics of the two $\sigma_j$ are reported in Fig.6. We note that the $\sigma_b$ dynamic is an oscillation around its stationary point. With the small value of $z_{0j}$ and the attractive regime the variables $\sigma_j$ oscillates without large interaction with the other variables, and so we do not observe secondary peaks or any sort of intermode interference.
Figure 6: The parameters, stationary points and the initial conditions are the same in Fig.5. The two $\sigma_j$ start near their stationary value.

In Fig.7 the initial conditions differs from Fig.5 only for the initial values of the two $z_{0j}(0)$, which are much larger. In this plot we see that the dynamics for the two variables is very similar to that in Fig.5 except for the oscillation amplitude that are larger now. We can conclude that despite these strongly distorted initial conditions the system undergoes a strong attraction that regulates its dynamics and produces a qualitatively similar movement for a large amplitude of $z_{0j}$ oscillations. In Fig.8 we observe that with initial condition for the two $z_{0j}$ far from zero (the stationary value) we have a sizable energy exchange involving the variables $\sigma_j$. In particular $\sigma_f$ shows secondary higher frequency peaks due to nonlinear energy transfer to the corresponding mode.

5 Discussion

The simple variational ansatz proposed here, with all its obvious limits, provides a nice description of the collective dynamics of a mixture of boson and fermions. we find that when fermions and bosons attract each other, a model where the two clouds are both centered at the trap minimum is perfectly appropriate. In the
Figure 7: The parameters and the stationary points are the same in Fig.5. The initial conditions: $z_{0b}(0) = -5$, $z_{0f}(0) = 2$, $\sigma_b(0) = 21$, $\sigma_f(0) = 2$: both the $z_{0j}$ are set in a value very different from equilibrium.
Figure 8: The parameters are in Fig.3. The initial conditions and the stationary points are the same as in Fig.7. The $\sigma_j$ start from initial value near the stationary point.
repulsive regime instead a nice symmetry-broken solution arises when we allow for the Gaussian clouds to displace.

An important extension of the present theory shall involves anisotropic Gaussian in anisotropic traps, as often done in experiment.
References


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