Bogoliubov-De Gennes normal-modes analysis of a cylindrically symmetric Bose-Einstein condensate

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Abstract

This thesis analyzes the collective modes of a Bose-Einstein condensate trapped in a harmonic cylindrically symmetric potential well. We use the Gross-Pitaeskii equation to evaluate the mean-field ground state, and the Bogoliubov-De Gennes formalism to compute normal modes. We solve both problems numerically by means of a newly developed code based on standard numerical techniques. In particular, we devised a self-consistent Newton-Raphson algorithm for the ground-state computation and we used LAPACK libraries to solve the normal-mode problem. We discuss the agreement of our numerical calculation with recent experiments of an atomic BEC confined in a electromagnetic trap.

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1 Introduction

Bose-Einstein condensation was first observed in the mid-1990s with vapors of rubidium [1] and sodium [2]. To obtain a Bose-Einstein condensate (BEC), the gaseous sample must be so cold that the De-Broglie thermal length
\[ \lambda(T) = \sqrt{\frac{2\pi\hbar^2}{mk_bT}} \]
must exceed the average spacing between atoms. For a gas of free non-interacting bosons, this implies the following condition on density and temperature:
\[ \lambda(T)^2 \frac{N}{V} > \zeta(3/2) \approx 2.6, \tag{1} \]
where
\[ \zeta(\alpha) = \sum_{n=0}^{\infty} \frac{1}{n^\alpha} \]
is the Riemann function. In practice, these extreme conditions of high density and low temperature are quite difficult to achieve experimentally because, if bosons are too closely packed together, they interact strongly and usually turn into the solid state. To reach the BEC state, an option is to obtain an extremely cold gas (i.e. \( T \sim nK \)), but relatively dilute, so that interatomic interactions are weak and well understood. These conditions, as discussed in [1], are achieved by performing the following operations:

- First, the atomic vapors are optically pre-trapped with a laser in a high-vacuum glass cell.
- Then a magnetic trap is created with an orbiting spherical magnetic quadrupole field plus a small uniform rotating transverse magnetic field both generated by coils located outside the vacuum cell. The result is an approximately harmonic potential acting on individual atoms and characterized by cylindrical symmetry such as
  \[ V_{\text{ext}}(r) = \frac{1}{2}m\omega_r^2 r^2 + \frac{1}{2}m\omega_z^2 z^2. \tag{2} \]
  where \( m \) is the atomic mass and \( \omega_z \) and \( \omega_r \) are the harmonic frequencies.
- Finally, the atoms are cooled evaporatively by applying a radiofrequency field. This frequency selectively drives out of the trap the atoms with higher energy, which populate the trap regions with higher magnetic field, by shifting their spin-flip transition frequencies upward by the Zeeman effect. Then this frequency is ramped downward adiabatically selecting gradually atoms with lower energy in an untrapped spin state until the sample reaches the desired temperature, which is a monotonic function of the radio frequency.
The spatial distribution of the cloud is measured from the absorption of a circularly polarized laser resonant with some well defined alkali transition line (for $^{87}\text{Rb}$ is $5S_{1/2} \rightarrow 5P_{3/2}$) and through this measurement, it is possible to acquire a 2D image of the velocity distribution. The main evidence of condensation is given by the fact that the central region of the velocity distribution displays the same axial symmetry as the potential, while the non-condensed fraction is thermally distributed and therefore isotropic. The experimental production of condensates has stimulated theoretical investigation for a deeper understanding of both the equilibrium state and the excitations of such systems.

Important features, which allow us to study the dynamic properties of the BECs, are the normal excitation frequencies that can be investigated \[3\] by adding a sinusoidal current in the coils that generate the rotating transverse field.

Section 2.2 summarises the theory for the ground state and excitations of BECs. In Section 3 we describe the computational implementation of two codes designed to evaluate respectively the ground state and the normal modes of a BEC in a cylindrically symmetric trap. In Section 4 we will report and discuss our results and analyze the agreement with experiment 3.

2 The model

Real-life atomic bosons are interacting, and this fact must be taken into account in the description of a BEC which differs substantially from a non-interacting gas. The grand canonical many-body Hamiltonian which describes $N$ interacting bosons trapped by an external potential is \[4, 5\]:

\[
\hat{K} = \hat{H} - \mu \hat{\mathcal{N}} = \int dr \hat{\Psi}^\dagger(r) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r)\right] \hat{\Psi}(r)
+ \frac{1}{2} \int dr dr' \hat{\Psi}^\dagger(r) \hat{\Psi}(r') V(r - r') \hat{\Psi}^\dagger(r') \hat{\Psi}(r)
- \mu \int dr \hat{\Psi}^\dagger(r) \hat{\Psi}(r), \tag{3}
\]

where $V(r - r')$ is the two-body interatomic potential and $\hat{\Psi}^\dagger(r)$ and $\hat{\Psi}(r)$ are the boson field operators that annihilate and create a particle at position $r$, and
which satisfy the following relations:

\[
\int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) = \hat{N}
\]

\[
\left[ \hat{\Psi}(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r}) \right] = \delta(\mathbf{r} - \mathbf{r}')
\]

\[
\left[ \hat{\Psi}(\mathbf{r}'), \hat{\Psi}(\mathbf{r}) \right] = \left[ \hat{\Psi}^\dagger(\mathbf{r}'), \hat{\Psi}^\dagger(\mathbf{r}) \right] = 0.
\] (4)

To avoid dealing with the many body Hamiltonian (3), which would require heavy numerical calculations, it is usually convenient to opt for a mean-field approach. In particular, \( \hat{\Psi}^\dagger(\mathbf{r}) \) and \( \hat{\Psi}(\mathbf{r}) \) can be rewritten in terms of Fock bosonic single-particle creation and annihilation operators \( \hat{a}^\dagger_\alpha \) and \( \hat{a}_\alpha \) and their respective single-particle wavefunctions \( \Psi_\alpha(\mathbf{r}) \):

\[
\hat{\Psi}^\dagger(\mathbf{r}) = \sum_\alpha \Psi_\alpha^\dagger(\mathbf{r}) \hat{a}^\dagger_\alpha, \quad \hat{\Psi}(\mathbf{r}) = \sum_\alpha \Psi_\alpha(\mathbf{r}) \hat{a}_\alpha,
\] (5)

where

\[
\hat{a}_\alpha |n_0, n_1, \ldots, n_\alpha, \ldots\rangle = \sqrt{n_\alpha + 1} |n_0, n_1, \ldots, n_\alpha + 1, \ldots\rangle,
\] (6)

\[
\hat{a}_\alpha |n_0, n_1, \ldots, n_\alpha, \ldots\rangle = \sqrt{n_\alpha} |n_0, n_1, \ldots, n_\alpha - 1, \ldots\rangle,
\] (7)

and \( \hat{n}_\alpha = \hat{a}_\alpha^\dagger \hat{a}_\alpha \) is the number operator that counts the particles in the single-particle quantum state \( |\alpha\rangle \) described by the wavefunction \( \Psi_\alpha(\mathbf{r}) \). Now, similarly to the non-interacting case, we analyze separately the contribution of the particles in the ground state \( |0\rangle \). Indeed, if eq. (11) is satisfied, one single-particle state can give a macroscopic contribution to the total number of particles \( N \). In other words, \( \frac{N_0}{N} \) is finite at thermodynamic limit. Operationally, this means that we consider \( \hat{a}_0^\dagger \) and \( \hat{a}_0 \) separately in the sum (11) and we consider them as numbers (and not operators) such as \( a_0^\dagger = a_0 = \sqrt{N_0} \). As a consequence, we can write the Bogoliubov approximation:

\[
\hat{\Psi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \hat{\Psi}'(\mathbf{r}, t),
\] (8)

where:

- \( \Phi(\mathbf{r}, t) \) is a complex function that fixes the mean value of the field operator \( \hat{\Psi}(\mathbf{r}, t) \), namely \( \Phi(\mathbf{r}, t) = \langle \hat{\Psi}(\mathbf{r}, t) \rangle \). \( \Phi \) is often named "condensate wave function”,

- \( \int |\Phi(\mathbf{r}, t)|^2 d\mathbf{r} = N_0 \), which indicates the mean number of particles of the condensate,

- \( \hat{\Psi}'(\mathbf{r}, t) \) is the purely fluctuating (\( \langle \hat{\Psi}'(\mathbf{r}, t) \rangle \equiv 0 \)) field operator that accounts for the depletion of the condensate. It is often named "deviation operator".
2.1 Ground State: The Gross-Pitaevskii Equation

In order to derive an equation for the condensate wave function, we write the time evolution of the field operator using the Heisenberg representation:

\[ i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = [\hat{\Psi}, \hat{H}] \]

\[ = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + \int d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}', t)V(\mathbf{r} - \mathbf{r}')\hat{\Psi}(\mathbf{r}', t) \right] \hat{\Psi}(\mathbf{r}, t). \]  

(9)

To simplify the equation above we must do some approximation on the interacting potential. We know [6] that the \( s \)-wave scattering amplitude is \( f_0(k) = \frac{1}{k} (\cot \delta(k) - i) \) and that for \( k \to 0 \), \( f_0(k) \to -a_s \) where \( a_s \) is a finite quantity called "\( s \)-wave scattering length". Given that only low-energy binary collisions take place in a dilute and extremely cold gas, \( a_s \) is the only parameter that really characterizes the interaction. As a consequence, we can write the interaction potential in eq. (10) as:

\[ V(\mathbf{r} - \mathbf{r'}) = g\delta(\mathbf{r} - \mathbf{r'}), \]

(10)

where \( g = \frac{4\pi \hbar^2 a_s}{m} \) is called the coupling constant. By the same token, we can replace \( \hat{\Psi}(\mathbf{r}, t) \) with \( \Phi(\mathbf{r}, t) \) in eq. (10) obtaining some sort of non-trivial mean-field "zero-th-order approximation" of the condensate time-evolution. Thus eq. (10) becomes:

\[ i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\Phi(\mathbf{r}, t)|^2 \right] \Phi(\mathbf{r}, t), \]

(11)

which is a time-dependent non-linear Schrödinger equation (NLSE). In particular we look for the stationary solution of this NLSE, namely a form such as:

\[ \Phi(\mathbf{r}, t) = e^{-\frac{i\mu}{\hbar} t} \phi(\mathbf{r}), \]

(12)

where:

- \( \mu \) is the chemical potential,
- \( \phi(\mathbf{r}) \) is a real function,
- \( \int d\mathbf{r} |\phi(\mathbf{r})|^2 = N_0 \), namely \( \phi \) is normalized to the total number of particle in the condensate.

Thus, by substituting eq. (12) in the eq. (11), we obtain:

\[ \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\phi(\mathbf{r})|^2 \right] \phi(\mathbf{r}) = \mu \phi(\mathbf{r}), \]

(13)
which is the celebrated Gross-Pitaevskii equation (GPE), namely the one that we solve numerically to compute the condensate ground state.

2.2 Normal modes: Bogoliubov Equations

To evaluate the normal modes of the condensate we must take a step back and substitute eq. (8) into the expression of the many-body Hamiltonian (3) and neglect terms higher than quadratic in \( \hat{\Psi}'(r) \). This yields:

\[
\hat{K} = \int dr \Phi^*(r) \left[ H_0 - \mu + \frac{1}{2}g|\Phi(r)|^2 \right] \Phi(r) + \int dr \Phi^*(r) \left[ H_0 - \mu + g|\Phi(r)|^2 \right] \hat{\Psi}'(r) + \int dr \hat{\Psi}^{\dagger}(r) \left[ H_0 - \mu + 2g|\Phi(r)|^2 \right] \hat{\Psi}'(r) + \frac{1}{2} \int dr \hat{\Psi}^{\dagger}(r) [\Phi(r)]^2 \hat{\Psi}'(r) + \frac{1}{2} \int dr \hat{\Psi}^{\dagger}(r) [\Phi^*(r)]^2 \hat{\Psi}'(r) \tag{14}
\]

where \( H_0 = -\frac{\hbar^2}{2m} \nabla^2 + V_{ext}(r) \) is the non-interacting Hamiltonian, the first term is a real number and that the second and the third terms vanish assuming that \( \mu \) and \( \Phi(r) \) satisfy eq. (13). Thus the grand canonical Bogoliubov-linearized Hamiltonian becomes:

\[
\hat{K}_B = c + \int dr \hat{\Psi}^{\dagger}(r) \left[ H_0 - \mu + 2g|\Phi(r)|^2 \right] \hat{\Psi}'(r) + \frac{1}{2} \int dr \hat{\Psi}^{\dagger}(r) [\Phi(r)]^2 \hat{\Psi}'(r) + \frac{1}{2} \int dr \hat{\Psi}^{\dagger}(r) [\Phi^*(r)]^2 \hat{\Psi}'(r), \tag{15}
\]

where \( c \) is a real number.

The Bogoliubov Hamiltonian can now be cast into the form of a collection of non-interacting quasiparticles. The concept of quasiparticle can be roughly understood as the combination of a particle with its influence on its local environment. In this way we can switch from an interacting to a fictitious non-interacting problem where the Hamiltonian can be written in form of sum of products of annihilation and creation operators like for example a 3D-harmonic oscillator Hamiltonian:

\[
\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + \sum_{i \in \{1,2,3\}} \frac{m\omega_i^2 \hat{x}_i^2}{2} = \sum_{i \in \{1,2,3\}} \hbar \omega_i \left( \hat{a}_i^{\dagger} \hat{a}_i + \frac{1}{2} \right). \tag{16}
\]
To do this, we expand the field operators of the non-condensed part as:

\[
\hat{\Psi}'(r) = \sum_{\alpha} \left[ u_{\alpha}(r) \hat{\beta}_\alpha + v_{\alpha}^*(r) \hat{\beta}_\alpha^\dagger \right], \quad \hat{\Psi}^\dagger(r) = \sum_{\alpha} \left[ u_{\alpha}^*(r) \hat{\beta}_\alpha^\dagger + v_{\alpha}(r) \hat{\beta}_\alpha \right],
\]  

where \(\hat{\beta}_\alpha\) and \(\hat{\beta}_\alpha^\dagger\) are the quasi-particle annihilation and creation operators in the state \(|\alpha\rangle\) which satisfy the usual boson commutation relations

\[
\left[ \hat{\beta}_\alpha, \hat{\beta}_\alpha^\dagger \right] = \delta_{\alpha\alpha'}, \quad \left[ \hat{\beta}_\alpha, \hat{\beta}_\alpha' \right] = \left[ \hat{\beta}_\alpha^\dagger, \hat{\beta}_\alpha'^\dagger \right] = 0,
\]  

and \(u_{\alpha}(r)\) \(v_{\alpha}(r)\) are square integrable wavefunctions. In particular, since \(\langle \hat{\Psi}' \rangle\) vanishes for \(N \to \infty\), in the thermodynamic limit we have \([\hat{\Psi}'(r'), \hat{\Psi}^\dagger(r)] \simeq \delta(r - r')\) (as in ref. [7]). By relations (18), we obtain the following normalization conditions on \(u_{\alpha}\) and \(v_{\alpha}\):

\[
\sum_{\alpha} \int \left[ u_{\alpha}(r) u_{\alpha}'(r') - v_{\alpha}(r') v_{\alpha}^*(r) \right] dr dr' = \delta(r - r'),
\]

\[
\int \left[ u_{\alpha}(r) u_{\alpha}'(r) - v_{\alpha}(r) v_{\alpha}^*(r) \right] dr = \delta_{\alpha\alpha'}.
\]

Moreover, if we impose that the Bogoliubov-approximate Hamiltonian can be written, to within a real number, as:

\[
\hat{K}_B = \sum_{\alpha} \hbar \omega_{\alpha} \hat{\beta}_\alpha^\dagger \hat{\beta}_\alpha,
\]

then we get the following set of coupled equations to be satisfied by \(u_{\alpha}\) and \(v_{\alpha}\):

\[
[H_0 - \mu + 2g|\Phi(r)|^2] u_{\alpha}(r) + g |\Phi(r)|^2 v_{\alpha}(r) = \hbar \omega_{\alpha} u_{\alpha},
\]

\[
[H_0 - \mu + 2g|\Phi(r)|^2] v_{\alpha}(r) + g |\Phi^*(r)|^2 u_{\alpha}(r) = -\hbar \omega_{\alpha} v_{\alpha},
\]

which are called Bogoliubov equations. These equations could be obtained also by linearizing eq.(11) searching for solutions of the form

\[
\Phi(r, t) = e^{-\frac{it\mu}{\hbar}} \left[ \phi(r) + u(r)e^{-i\omega t} + v^*(r)e^{i\omega t} \right].
\]

Equations (22) and (23) can be rearranged in matrix form as a generalized eigenvalue problem by defining the following quantities:

\[
\mathcal{L}_{\alpha,\beta} = \langle \alpha | H_0 - \mu + 2g|\Phi(r)|^2 |\beta\rangle,
\]

\[
\mathcal{M}_{\alpha,\beta} = \langle \alpha | g |\Phi(r)|^2 |\beta\rangle,
\]

\[
E = \hbar \omega,
\]

\[
\chi_{\alpha} = \begin{pmatrix} u_{\alpha} \\ v_{\alpha} \end{pmatrix}.
\]
Now we can formally write eqs. (22) and (23) as a matrix equation where $H_B$ is the Bogoliubov Hamiltonian:

$$H_B \chi_\alpha = \Sigma_3 \hbar \omega_\alpha \chi_\alpha,$$

(26)

where:

$$H_B = \begin{pmatrix} \mathcal{L} & g(\Phi(r))^2 \\ g(\Phi^*(r))^2 & \mathcal{L} \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

(27)

where $I$ is the identity operator. The size of this generalized eigenvalue problem is twice the number of basis functions $\varphi_\lambda$ used to expand the field operator.

The linear problem defined by eqs. (26) and (27) is to be solved numerically as described in the following section.

### 3 Technical implementation

Defining $\phi_g(r) \equiv \phi(r)/\sqrt{N_0}$, so that $\phi_g(r)$ is normalized to unity, we can reformulate eqs. (13), (22), and (23) as follows:

$$[(H_0 - \mu + \hbar \omega_\alpha) + 2gN_0|\phi_g(r)|^2] v_\alpha(r) + gN_0 [\phi_g^*(r)]^2 u_\alpha(r) = 0,$$

(29)

$$[(H_0 - \mu + \hbar \omega_\alpha) + 2gN_0|\phi_g(r)|^2] u_\alpha(r) + gN_0 [\phi_g^*(r)]^2 v_\alpha(r) = 0.$$  

(30)

This form is more convenient from the numerical point of view because it involves explicitly the number of bosons in the condensate $N_0$.

### 3.1 Solving the Gross-Pitaevskii Equation

Given the Gross-Pitaevskii Hamiltonian in the cylindrically symmetric harmonic trap,

$$\hat{H}_{GP} = \hat{H}_0 + \hat{H}_{int}$$

$$= \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_r^2 r^2 + \frac{1}{2} m \omega_z^2 z^2 \right] + gN_0|\phi_g(r)|^2,$$

(31)

the approach we use is expanding the ground-state function $\phi_g(r)$ on the well-known basis of eigenfunctions of the non-interacting Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_r^2 r^2 + \frac{1}{2} m \omega_z^2 z^2.$$
Indeed this standard quantum mechanics problem can be separated into a two-dimensional harmonic oscillator in the \((x, y)\)-plane, where we use polar coordinates \((r, \theta)\) plus a one-dimensional harmonic oscillator along the \(z\)-axis. These two problems correspond to the following stationary Schrödinger equations [11]:

\[
-\frac{\hbar^2}{2m} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] + \frac{1}{2} \omega_r^2 r^2 U(r, \theta) = \varepsilon \perp U(r, \theta) \quad (33)
\]
\[
-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{1}{2} \omega_z^2 z^2 Z(z) = \varepsilon_z Z(z) \quad (34)
\]

Relatively to eq. (33), since \([\hat{H}_0, \hat{L}_z] = 0\), we can take \(m_z = \langle \hat{L}_z \rangle / \hbar = 0, \pm 1, \pm 2, \ldots\) (where \(\hat{L}_z\) is the angular momentum operator along the \(z\)-axis) as a good quantum number and factorize \(U(r, \theta) = R(r) \Theta(\theta)\). Accordingly we obtain the following normalized eigenfunctions:

\[
\Theta_{m_z}(\theta) = \frac{e^{im_z \theta}}{\sqrt{2\pi}}, \quad m_z = 0, \pm 1, \pm 2, \ldots
\]
\[
R_{nr}^{[m_z]}(r) = \left[ \frac{2\alpha_r^2 n_r!}{(n_r + |m_z|)!} \right]^{1/2} (\alpha_r r)^{n_r} e^{-\frac{1}{2} \alpha_r^2 r^2} L_{nr}^{[m_z]}(\alpha_r^2 r^2), \quad n_r = 0, 1, 2, \ldots
\]
\[
Z_{nz}(z) = \left[ \frac{\alpha_z}{\pi^{1/2} 2^{n_z} (n_z)!} \right]^{1/2} e^{-\frac{1}{2} \alpha_z^2 z^2} H_{n_z}(\alpha_z z), \quad n_z = 0, 1, 2, \ldots
\]

where:

- \(\alpha_x, \alpha_z = \frac{m_x, m_z}{\hbar}\) are the inverse characteristic lengths of the harmonic oscillator,
- \(H_{n_z}\) is the Hermite polynomial [9] of degree \(n_z\),
- \(L_{nr}^{[m_z]}\) is the generalized Laguerre polynomial [9] of degree \(n_r\) and order \(|m_z|\).

The corresponding eigenvalues are:

\[
\varepsilon \perp = \hbar \omega_r (2n_r + |m_z| + 1), \quad (35)
\]
\[
\varepsilon_z = \hbar \omega_z (n_z + \frac{1}{2}). \quad (36)
\]

Collecting the pieces together, the energy eigenvalue and the corresponding eigenfunction relative to the state \(|\lambda\rangle = |n_r, n_z, m_z\rangle\) are:

\[
\varepsilon_\lambda = \varepsilon \perp + \varepsilon_z = \hbar \omega_r (2n_r + |m_z| + 1) + \hbar \omega_z (n_z + \frac{1}{2}). \quad (37)
\]
\[
\varphi_\lambda(r) = R_{nr}^{[m_z]}(r) \Theta_{m_z}(\theta) Z_{nz}(z). \quad (38)
\]
To solve the interacting GP problem we expand the ground state function \( \phi_g(r) \) as a linear combination of eigenfunctions of \( H_0 \) as follows:

\[
\phi_g(r) = \sum_\lambda a_\lambda \varphi_\lambda(r),
\]

where

\[
a_\lambda = \langle \lambda | \phi_g \rangle = \int \varphi_\lambda^*(r) \phi_g(r) dr
\]

are the unknown complex coefficients defining \( \phi_g \). To do this, we choose a certain cutoff to keep a finite number \( N_b \) of states in the basis, by choosing the maximum value \( N_{\text{max}} \) of the number operator \( \hat{N} \) relative of the non-interacting Hamiltonian \( \hat{H}_0 \) that satisfies the following condition:

\[
\hat{N}|n_r, n_z, m_z\rangle = (2n_r + |m_z| + n_z)|n_r, n_z, m_z\rangle.
\]

Now we can rewrite the Gross-Pitaevskii equation on the truncated basis of \( N_b \) harmonic eigenfunctions by plugging eq. (39) in eq. (28) and projecting it on a generic \( |\lambda\rangle \) state. In this way we can define a non-linear \( N_b \)-component function whose \( \lambda \)-th component is

\[
F_\lambda(a, \mu) \equiv (\varepsilon_\lambda - \mu)a_\lambda + gN_0 \sum_{\lambda_1=1}^{N_b} \sum_{\lambda_2=1}^{N_b} \sum_{\lambda_3=1}^{N_b} a_{\lambda_1}^* a_{\lambda_2} a_{\lambda_3} C(\lambda, \lambda_1, \lambda_2, \lambda_3),
\]

where \( C(\lambda, \lambda_1, \lambda_2, \lambda_3) \) is an overlap integral of four functions that comes from the non-linear part of eq. (28). Its explicit form is:

\[
C(\lambda, \lambda_1, \lambda_2, \lambda_3) = \int \varphi_\lambda^*(r) \varphi_{\lambda_1}^*(r) \varphi_{\lambda_2}(r) \varphi_{\lambda_3}(r) dr.
\]

In this way the Gross-Pitaevskii equation is reduced to a non-linear system expressed in concise vector form as:

\[
F(a, \mu) = 0.
\]

Here \( F : C^{N_b+1} \rightarrow C^{N_b} \) is the non-linear \( N_b \)-dimensional function of \( N_b+1 \) variables defined by eq. (54). The overlap factor can be further written out as:

\[
C(\lambda, \lambda_1, \lambda_2, \lambda_3) = A_\lambda A_{\lambda_1} A_{\lambda_2} A_{\lambda_3} \Xi(n_z, n_{z_1}, n_{z_2}, n_{z_3}) \times \Upsilon(\rho, \rho_1, \rho_2, \rho_3) K(m_z, m_{z_1}, m_{z_2}, m_{z_3})
\]
where $\rho_i = (n_{r_i}, m_{z_i})$. We make the substitutions $\xi = \alpha z$ and $\zeta = \alpha^2 r^2$ and express the integrals as:

\[
\Upsilon(\rho, \rho_1, \rho_2, \rho_3) = \frac{1}{2\alpha^2} \int_0^\infty \frac{\zeta^{1/2}(m_z + m_{z_1} + m_{z_2} + m_{z_3})}{\zeta^{1/2}} e^{-2\zeta} d\zeta \quad (46)
\]

\[
\Xi(n_z, n_{z_1}, n_{z_2}, n_{z_3}) = \frac{1}{\alpha z} \int_{-\infty}^{\infty} e^{-2\xi} H_{n_z}(\xi) H_{n_{z_1}}(\xi) H_{n_{z_2}}(\xi) H_{n_{z_3}}(\xi) d\xi \quad (47)
\]

\[
K(m_z, m_{z_1}, m_{z_2}, m_{z_3}) = \int_0^{2\pi} e^{-i(m_z + m_{z_1} - m_{z_2} - m_{z_3})\theta} d\theta = 2\pi \delta_{m_z + m_{z_1}, m_{z_2} + m_{z_3}} (49)
\]

\[
A_{\lambda_i} = \frac{1}{\sqrt{2\pi}} \left[ \frac{2\alpha_{\lambda_i}^2 n_{r_i}}{(n_{r_i} + |m_{z_i}|)!} \right]^{1/2} \left[ \frac{\alpha_{\lambda_i}}{\pi^{1/2} 2^{n_{r_i}} n_{z_i}} \right]^{1/2}. \quad (50)
\]

The integrals (47) and (48) are evaluated by the Gaussian Quadrature method [10], as explained in Appendix A.

Solutions of the system (44) can be found through the Newton-Raphson method [10]. The idea behind this method is to linearize the problem locally by writing:

\[
F(a + \delta a) = F(a) + J \cdot \delta a + O(\delta a^2), \quad (51)
\]

and setting $F(a + \delta a) = 0$, which allows one to find the step $\delta a$ by solving a linear system:

\[
J \cdot \delta a = -F(a). \quad (52)
\]

Then, by starting from an initial guess $a^{(0)}_\lambda$, one proceeds by adding the step $\delta a$ found through the linear system (52) and use $a^{(1)}_\lambda = a^{(0)}_\lambda + \delta a$ as the new starting point. This procedure is iterated until one finds an ”equilibrium vector” $a^*$, such that $F(a^*) = 0$ is satisfied within a given tolerance.

The problem, however, is slightly more complicated because the final stationary state found at the end of this iterative procedure could happen to be some excited eigenstate of $H_{GP}$. To overcome this problem and ensure that the final solution is indeed the state with lowest energy, we choose as a starting point the solution of the non-interacting ground state, namely $|\lambda_0\rangle = |n_r = 0, n_z = 0, m_z = 0\rangle$.

In practice we take for the starting vector of coefficients and chemical potential:

\[
a^{(0)} = (1, 0, 0, 0, ..., 0),
\]

\[
\mu^{(0)} = \varepsilon_0 = \hbar \omega_r + \frac{1}{2} \hbar \omega_z, \quad (53)
\]

and then find an iterative solution $a(N_0^{(1)})$ only for a small number of particles that interact weakly and does not differ much from the non-interacting solution.
Then we raise the number of particles $N_0^{(1)} \to N_0^{(2)}$ and choose as a new starting point the solution of the previous step. At the end of this "adiabatic expansion", if the basis size is sufficiently large, we determine the ground state and its corresponding chemical potential for many different numbers of particles of the condensate, and in particular for the required $N_0^{max}$.

Nevertheless this is not the only difficulty because $F$ is also a function of $\mu$, which is itself a function of the vector of coefficients $a$ and of the number of particles in the condensate $N_0$:

$$
\mu(a, N_0) = \langle \phi_y | H_{GP} | \phi_y \rangle = \sum_{\lambda=0}^{N_0-1} \varepsilon_\lambda |a_\lambda|^2 
+ g N_0 \sum_{\lambda=1}^{N_0} \sum_{\lambda_1=1}^{N_0} \sum_{\lambda_2=1}^{N_0} \sum_{\lambda_3=1}^{N_0} a_{\lambda_1}^* a_{\lambda_2} a_{\lambda_3} C(\lambda, \lambda_1, \lambda_2, \lambda_3),
$$

(54)

Thus, when we find an iterative solution $a(N_0^{new})$ starting from, say, $N_0^{old}$, we use $\mu^{old}$, which is a trial quantity itself for the new number of particles $N_0^{new}$. This means that, in order to find the real solution of eq. (44) for a given number of particles $N_0$, we must also adjust iteratively $\mu^{new}$ until the true self-consistent $\mu$ and $a$ is found. The iterative procedure to find a self-consistent solution for a given $N_0^{max}$ is described schematically in figure 1.

Some further considerations are in order regarding the search of the ground state by means of this self-consistent iteration. Indeed the choice of $|\lambda_0\rangle = |n_r = 0, n_z = 0, m_z = 0\rangle$ as the starting state has remarkable consequences:

- Since $[\hat{H}_0, \hat{L}_z] = 0$, then $\langle \lambda | \hat{H}_0 | \lambda \rangle \propto \delta_{m_z,m_z}$. Hence, by choosing as initial guess an $m_z = 0$ state, we are restricted to the $m_z = 0$ Hilbert subspace. Only the states belonging to this subspace will effectively have a non-zero coefficient $a_\lambda$ in the final ground-state decomposition. This means that it is sufficient to include only $m_z = 0$ states in the basis. Moreover, this restriction makes the eigenfunctions real because $\Theta_0(\theta) = \frac{1}{\sqrt{2\pi}}$. Hence the ground-state vector of coefficients $a$ turns out to be real, so it is simpler to compute it numerically. Moreover, the generic eigenvalue associated to the eigenstate $|\lambda\rangle = |n_r, n_z\rangle$, is:

$$
\varepsilon_\lambda = \hbar \omega_r (2n_r + 1) + \hbar \omega_z (n_z + \frac{1}{2}),
$$

(55)

and, as a consequence, the non-interacting number operator satisfies the following equation:

$$
\hat{N}|n_r, n_z\rangle = (2n_r + n_z)|n_r, n_z\rangle.
$$

(56)
Start with \( N_0^{\text{old}} = 1 \) and initial guess \( a^{(0)} \), \( \mu^{(0)} \).

Compute the new chemical potential \( \mu^{\text{new}} \),

Stop the iteration

Yes

No

Solve the non-linear system using initial guess \( a^{\text{old}} \) and \( \mu^{\text{old}} \) and find the vector of coefficients \( a^{\text{new}} \).

Compute the new chemical potential \( \mu^{\text{new}} \)

Yes

No

\( N_0^{\text{new}} > N_0^{\text{old}} \) ?

\( a^{\text{old}} = a^{\text{new}} \)

\( \mu^{\text{old}} = \mu^{\text{new}} \)

\( N_0 = N_0^{\text{max}} \) ?

No

Newt-Raphs

\( \| a^{\text{new}} - a^{\text{old}} \| < \text{tol} \) or \( |\mu^{\text{new}} - \mu^{\text{old}}| < \text{tol} \) ?

Yes

\( a^{\text{old}} = a^{\text{new}} \) and \( \mu^{\text{old}} = \mu^{\text{new}} \)

Figure 1: Iterative procedure for the solution of the Gross-Pitaevskii equation on the basis of the non-interacting wavefunctions for a given number of condensed bosons \( N_0^{\text{max}} \) within a given tolerance.
Thus, given a certain cutoff $N_{\text{max}}$, the states involved in the decomposition are:

\[ N = 0 \quad \rightarrow \quad |n_z = 0, n_r = 0 \rangle \\
N = 1 \quad \rightarrow \quad |n_z = 1, n_r = 0 \rangle \\
N = 2 \quad \rightarrow \quad |n_z = 2, n_r = 0 \rangle, \quad |n_z = 0, n_r = 1 \rangle \\
N = 3 \quad \rightarrow \quad |n_z = 3, n_r = 0 \rangle, \quad |n_z = 1, n_r = 1 \rangle \\
\vdots \\
N = N_{\text{max}} \quad \rightarrow \quad |n_z = N_{\text{max}}, n_r = 0 \rangle, \quad \cdots \quad |n_z = \text{mod}(N_{\text{max}}, 2), n_r = \left[ \frac{N_{\text{max}}}{2} \right] \rangle \]  

(57)

where $\left[ \cdot \right]$ stands for integer part and every set corresponding to a certain $N$ has $s(N) = \left[ \frac{N}{2} \right] + 1$ states. By fixing the cutoff $N_{\text{max}}$, we choose the size of the basis

\[ N_b(N_{\text{max}}) = \sum_{i=0}^{N_{\text{max}}} s(N) = \sum_{i=0}^{N_{\text{max}}} \left( \left[ \frac{i}{2} \right] + 1 \right) = (N_{\text{max}}+1) + \left[ \frac{N_{\text{max}}}{2} \right] \left[ \frac{N_{\text{max}}+1}{2} \right]. \]

(58)

- We know that $[\hat{H}_0, \hat{\sigma}_z] = 0$ where $\hat{\sigma}_z$ is the operator reflecting across the $(x, y)$-plane. This means that $\hat{H}_0$ and $\hat{\sigma}_z$ have a common set of eigenstates. Note that in the basis state $|\lambda \rangle = |n_r, n_z \rangle$ we have:

\[ \hat{\sigma}_z |n_z, n_r \rangle = (-1)^{n_z} |n_z, n_r \rangle. \]  

(59)

Accordingly, in the $(m_z = 0)$-subspace, the global $\hat{\sigma}_z$ character of a basis state $|\lambda \rangle$ is determined exclusively by the quantum number $n_z$. Then, by choosing as initial guess the non-interacting ground state with $n_z = 0$, we are restricted to the Hilbert subspace with even $n_z$. This restriction is mathematically expressed by the overlap integral $\Xi(n_z, n_{z_1}, n_{z_2}, n_{z_3})$ of eq. (48) that satisfies the following condition:

\[ \Xi(n_z, n_{z_1}, n_{z_2}, n_{z_3}) = 0 \text{ if } (n_z + n_{z_1} + n_{z_2} + n_{z_3}) \text{ is odd.} \]  

(60)

These considerations imply a remarkable saving of computational effort because we know in advance which eigenfunctions will have a non-zero coefficient $a_\lambda$ in the final decomposition of the ground state $\phi_g(r) = \sum_\lambda a_\lambda \varphi_\lambda(r)$, so that we exclude them from the start. Operationally this implies the exclusion of all the states corresponding to a $N$ odd in the classification of eq. (57).

Once we have a self-consistent solution for the coefficients $a$ (and therefore for the wavefunction $\phi_g(r)$ of eq. (25)), with the corresponding $\mu$, we have all the ingredients necessary to proceed to the normal-mode computation.
3.2 Solving the Bogoliubov-De Gennes Equations

Following an approach completely analogous to that of the previous section, we can also expand the Bogoliubov functions \( u_\alpha(r) \) and \( v_\alpha(r) \) on the truncated basis:

\[
\begin{align*}
    u_\alpha(r) &= \sum_\lambda b_\lambda^{(\alpha)} \varphi_\lambda(r), \\
    v_\alpha(r) &= \sum_\lambda c_\lambda^{(\alpha)} \varphi_\lambda(r),
\end{align*}
\]

(61)

where \( \alpha \) and \( \lambda \in \{1, ..., N_b\} \). Plugging these expansions in the Bogoliubov equations (29) and (30) and projecting them on a generic state \( \langle \lambda' | \) yields the following system of coupled equations:

\[
\begin{align*}
    b_\lambda^{(\alpha)} [\varepsilon_\lambda - E_\alpha - \mu] + 2gN_0 \sum_\lambda b_\lambda^{(\alpha)} w_{\lambda,\lambda'} + gN_0 \sum_\lambda c_\lambda^{(\alpha)} w_{\lambda,\lambda'} &= 0, \\
    c_\lambda^{(\alpha)} [\varepsilon_{\lambda'} + E_\alpha - \mu] + 2gN_0 \sum_\lambda c_\lambda^{(\alpha)} w_{\lambda,\lambda'} + gN_0 \sum_\lambda b_\lambda^{(\alpha)} w_{\lambda,\lambda'} &= 0,
\end{align*}
\]

(62, 63)

where \( w_{\lambda,\lambda'} \) is an overlap integral, whose explicit form, recalling eq. (39), is:

\[
\begin{align*}
    w_{\lambda,\lambda'} &= \int \varphi_{\lambda'}^*(r) |\phi_g(r)|^2 \varphi_\lambda(r) \, dr \\
                   &= \sum_{\lambda(1)=1}^{N_b} \sum_{\lambda(2)=1}^{N_b} a_{\lambda(1)}^* a_{\lambda(2)} \int \varphi_{\lambda'}^*(r) \varphi_{\lambda(1)}^*(r) \varphi_{\lambda(2)}(r) \varphi_\lambda(r) \, dr.
\end{align*}
\]

(64)

Here \( a \) is the ground-state vector of coefficients found numerically with the method described in the previous section. For simplicity we only compute \( m_z = 0 \) excitations, which means that we only include \( m_z = 0 \) basis functions in the BdG calculations, as in the GP solution. Note that writing \( |\phi_g(r)|^2 \) or \( (\phi_g(r))^2 \) is the same because \( |\phi_g\rangle \) is a real \( m_z = 0 \) state. Moreover, our basis eigenfunctions are real as well.

Some further considerations are needed regarding \( w_{\lambda,\lambda'} \). Since \( |\phi_g\rangle \) is a linear superposition of even states \( |\varphi_\lambda\rangle \), it is also even (we recall that the parity of a state is exclusively determined by the quantum number \( n_z \)). Then \( w_{\lambda,\lambda'} \) vanishes whenever \( |\varphi_\lambda\rangle \) and \( |\varphi_{\lambda'}\rangle \) are respectively even and odd. This means in particular that there are two different types of normal modes that do not couple with those of the other type: even modes formed by a linear superposition of even basis states and odd modes formed by odd basis states. These two types of normal modes live in uncoupled subspaces, and therefore it is possible to separate the Bogoliubov Hamiltonian into two different generalized eigenvalue problems. We
apply the procedure that follows separately to both problems. In particular for
the even normal-modes we use the same basis as for the ground-state decom-
position, namely the first \(N_b\) even basis eigenfunctions (corresponding to even \(N\)),
according to eq. (57). On the other hand, for the odd normal modes, we have two
possibilities: if we choose an even \(N_{\text{max}}\), we use a slightly smaller basis extended
to \(N_{\text{max}} - 1\), formed by the first \(N_{\text{b}}^{(\text{odd})} = N_b - s(N_{\text{max}})\) odd basis eigenfunctions,
corresponding to the states with odd \(N\) in eq. (57). Conversely, if we choose
an odd \(N_{\text{max}}\), we obtain an odd basis formed by the first \(N_{\text{b}}^{(\text{odd})}\) odd
basis eigenfunctions.

Now we can write a matrix representation of the Bogoliubov Hamiltonian
\(H_B\). It consist of a \(2N_b \times 2N_b\) (or \(2N_{\text{b}}^{(\text{odd})} \times 2N_{\text{b}}^{(\text{odd})}\)) matrix

\[
H_B = \begin{pmatrix}
\mathcal{L} & \mathcal{M} \\
\mathcal{M}^* & \mathcal{L}
\end{pmatrix},
\]

(65)

where:

\[
\mathcal{L} = \begin{pmatrix}
(\varepsilon_1 - \mu) + 2gN_0w_{1,1} & 2gN_0w_{1,2} & \cdots & 2gN_0w_{1,N_b} \\
2gN_0w_{2,1} & (\varepsilon_2 - \mu) + 2gN_0w_{2,2} & \cdots & 2gN_0w_{2,N_b} \\
\vdots & \vdots & \ddots & \vdots \\
2gN_0w_{N_b,1} & 2gN_0w_{N_b,2} & \cdots & (\varepsilon_{N_b} - \mu) + 2gN_0w_{N_b,N_b}
\end{pmatrix},
\]

(66)

and

\[
\mathcal{M} = \begin{pmatrix}
gN_0w_{1,1} & gN_0w_{1,2} & \cdots & gN_0w_{1,N_b} \\
gN_0w_{2,1} & gN_0w_{2,2} & \cdots & gN_0w_{2,N_b} \\
\vdots & \vdots & \ddots & \vdots \\
gN_0w_{N_b,1} & gN_0w_{N_b,2} & \cdots & gN_0w_{N_b,N_b}
\end{pmatrix}.
\]

(67)

As mentioned in the previous section, equations (62) and (63) can be written as
a generalized eigenvalue problem:

\[
\begin{pmatrix}
\mathcal{L} & \mathcal{M} \\
\mathcal{M}^* & \mathcal{L}
\end{pmatrix}
\begin{pmatrix}
u_\alpha \\
v_\alpha
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}
\begin{pmatrix}
u_\alpha \\
v_\alpha
\end{pmatrix} E_\alpha
\]

(68)

We can reduce this problem to a standard eigenvalues problem by multiplying
both sides of eq. (68) by \(\Sigma^{-1}_3\) (which is equal to \(\Sigma_3\) itself). Hence we obtain the
following equation:

\[
\begin{pmatrix}
\mathcal{L} & \mathcal{M} \\
-\mathcal{M}^* & -\mathcal{L}
\end{pmatrix}
\begin{pmatrix}
u_\alpha \\
v_\alpha
\end{pmatrix} = E_\alpha
\begin{pmatrix}
u_\alpha \\
v_\alpha
\end{pmatrix}.
\]

(69)
To solve numerically this eigenvalue problem, we use LAPACK library \[13\]. Since the matrix \((\Sigma_3^{-1}H_B)\) is not symmetric we evaluate only the right eigenvectors of \((\Sigma_3^{-1}H_B)\) and the associated eigenvalues.

4 Results

In this section we show the results of our numerical calculation applied to a trapped boson droplet whose physical parameters match the experimental setup of ref. \[3\] where:

\[
\begin{align*}
\omega_z &= 2345.84 \text{ Hz}, \\
\omega_r &= 829.38 \text{ Hz}, \\
\frac{\omega_z}{\omega_r} &= \sqrt{8}, \\
a_s &= 5.81 \cdot 10^{-9} \text{ m}, \\
N_0 &= 4500 \pm 300.
\end{align*}
\] (70)

In that experiment, after creating a condensate with the techniques recalled in section \[11\] the normal-modes are investigated by applying a small time-dependent perturbation to the coils that generate the transverse trap potential. By appropriately tuning the phases of the currents through the coils, two different types of perturbation are generated, respectively characterized by two different kinds of symmetry, \(m_z = 0\) and \(m_z = 2\). We focus exclusively on the experimental data relative to the \(m_z = 0\) symmetric perturbation. We use a dimensionless set of units, based on \(\hbar, a_z, \text{ and } \omega_z\) being taken as unity rather than \(\hbar = 1.05455 \cdot 10^{-34} \text{ J-s}, a_z = 5.58 \cdot 10^{-7} \text{ m} \) and \(\omega_z = 2345.84 \text{ Hz}\). Energies are therefore measured in units of \(\hbar \omega_z = 2.47 \cdot 10^{-31} \text{ J}\).

4.1 The condensate wavefunction

The ground-state calculation leads to the wavefunction displayed in figure \[2\]. Since the interaction is repulsive \((a_s > 0)\), as \(N_0\) grows the atoms are pushed outward and the central density becomes flatter, as shown in figures \[3\] and \[4\].

During the iterative procedure, it is interesting to evaluate how the self-consistent chemical potential \(\mu\) evolves as the number of particles in the condensate grows. It is useful to compare it with the chemical potential computed in the Thomas-Fermi (TF) approximation, \(\mu_{TF}\). This approximation consists in
Figure 2: Condensate spatial density $|\phi_g(r, z)|^2$ attained by solving the GPE with the parameters of eq. (70).

Figure 3: Condensate spatial density section in the radial direction with different numbers of condensate particles $N_0$. The solid curve ($N_0 = 4500$) matches the 3D Plot of 2.
Figure 4: Condensate spatial density section in the axial direction with different numbers of condensate particles $N_0$. 
neglecting the kinetic term in eq. (13) which, as $N_0$ grows, becomes less and less important with respect to the interaction and the potential energy. Within this approximation, the solution of eq. (13) has a simple analytic form, namely:

$$\phi_g^{TF}(r) = \begin{cases} \sqrt{\frac{1}{N_0g} [\mu - V_{\text{ext}}(r)]}, & \text{if } V_{\text{ext}}(r) \leq \mu, \\ 0, & \text{if } V_{\text{ext}}(r) > \mu. \end{cases}$$ \hspace{1cm} (71)$$

The normalization condition $\int |\phi_g(r)|^2 = 1$ provides an explicit relation between the chemical potential and the number of particles in the condensate:

$$\mu_{TF} = \frac{\hbar \omega_{ho}}{2} \left( \frac{15 N_0 a_s}{a_{ho}} \right)^{2/5}.$$ \hspace{1cm} (72)

where $a_{ho}$ and $\omega_{ho}$ are the geometric averages of the harmonic lengths and frequencies:

$$a_{ho} = (a_r a_r a_z)^{1/3},$$

$$\omega_{ho} = (\omega_r \omega_r \omega_z)^{1/3}. \hspace{1cm} (73)$$

This approximation is valid only for $\frac{N_0 a_s}{a_{ho}} \gg 1$; i.e., when the kinetic quantum pressure term is significant only at the boundary. Thus, as $N_0$ raises, the relative difference between the $\mu_{SC}$ and $\mu_{TF}$ shrinks, as shown by figure 5. Note that the self-consistent chemical potential starts from the non-interacting ground-state energy, as indicated in the figure. For example, for $N_0 = 10000$, in experiment [3] $\frac{N_0 a_s}{a_{ho}} \simeq 70$, therefore we are in the strong interacting limit as shown by figure 6 where the numerically calculated radial density almost coincides with the TF approximation. In these conditions, the relative error $|\mu - \mu_{TF}|/\mu \simeq 0.03$ is indeed very small.

### 4.2 Normal-mode analysis

The numerical calculations are based on the quasiparticle approximation, which is valid only for the lowest excitation modes. These modes refer to collective oscillations of the condensate, while the highest ones, for which our calculations become less reliable, refer mainly to the single-particle excitations. Thus we focus only on the lowest modes and compare our results with the experimental data.

It is insightful to compare the eigenfrequencies we find with the non-interacting excitation frequencies, namely:

$$\omega(n_r, n_z) = 2n_r \omega_r + n_z \omega_z.$$ \hspace{1cm} (74)
Figure 5: Comparison between $\mu_{SC}$, obtained by the numerical solution of the GPE, and the simple TF approximation $\mu_{TF}$ of eq. (72) as functions of $N_0$. 
Figure 6: Comparison between the radial section of $|\phi_g(r)|^2$, obtained by the numerical solution of the GPE, and the simple TF approximation $|\phi_{TF,g}(r)|^2$ of eq. 71 with $N_0 = 10000$. 
Figure 7: The ordered even $n_z$ eigenfrequencies of the interacting condensate for $N_0 = 4500$. The horizontal axis has progressive numbers in order of increasing frequency. The squares mark the non-interacting eigenfrequencies $\omega(n_r, n_z)$. 
From figures 7 and 8 it is clear that, because of interaction, the positive eigenfrequencies are shifted downward (for the negative ones the shift is opposite). This decrease is due mainly to the increase of the chemical potential (Fig. 5) with interaction. In the Bogoliubov Hamiltonian for the generalized eigenvalue problem of eq. (69), $\mu$ appears with a negative sign as is clear when eq. (69) is written as the sum of the unperturbed non-interacting Hamiltonian plus a non-linear term:

$$H_B = \begin{pmatrix} H_0 & 0 \\ 0 & -H_0 \end{pmatrix} + gN_0 \begin{pmatrix} -\frac{\mu(N_0)}{gN_0} + 2gN_0|\phi_g|^2 & gN_0|\phi_g|^2 \\ -gN_0|\phi_g|^2 & \frac{\mu(N_0)}{gN_0} - 2gN_0|\phi_g|^2 \end{pmatrix} \quad (75)$$

The reason for the downward (upward) shift of positive (negative) eigenfrequencies is the chemical potential that depends on the interaction, since it is a function of $N_0$ and has the behaviour displayed in figure 5. This negative (positive) shift is the same for all frequencies. On the other hand, the non-linear term $|\phi_g|^2$ is always positive but yields a different contribution for different states, and accounts for the loss of harmonicity, namely for the fact that the computed frequencies are not multiples or combinations of some “fundamental” ones, as it happens instead in the non-interacting case. Figures 9 and 10 display the detailed evolution of few lowest even and odd eigenfrequencies with the interaction strength. Note that in the non-interacting limit ($N_0 \rightarrow 0$ and $\mu(N_0) \rightarrow \mu(0)$) the lowest eigenfrequency becomes trivially $\omega(0,0) = 0$, which coincides with the solution of the
Figure 9: The four lowest even $n_z$ eigenfrequencies as functions of the number of condensate particles $N_0$. For every curve, the corresponding non-interacting frequency is indicated. Note the level crossing between the frequencies corresponding to $6\omega_r$ and $2\omega_z$. 
Figure 10: The four lowest odd $n_z$ eigenfrequencies as functions of the number of condensate particles $N_0$. For every curve the corresponding non-interacting frequency is indicated. Note that the purely dipole mode at $\omega = \omega_z$ does not change with interaction, indicating that it represents a rigid oscillation of the condensate inside the trap.
Gross-Pitaevskii equation. Therefore the interacting analogue of the trivial null frequency has not been considered in figures 7 and 9. Moreover, we can have a level crossing, as shown in figure 9.

\[ \omega_2 = \{ \omega_2 \} \]

Figure 11: Displacement functions \( u(r) + v(r) \) and \( u(r) - v(r) \) relative to the second odd normal mode of a BEC composed of \( N_0 = 4500 \) atoms.

Furthermore, it is interesting to address the behaviour of the lowest odd mode in figure 11, which turns out to be independent of the interaction strength. Indeed this mode requests a rigid oscillation of the condensate. The frequency independence on interaction is a consequence of the special dynamical property of the harmonic potential well, which is assumed for the trap [12]. Therefore, the lowest odd oscillation mode is purely axial, as one would expect since it refers to the dipole mode, which is a non-deformed axial displacement of the condensate at the same frequency of the non-interacting Hamiltonian, namely \( \omega_z \), corresponding
to the lowest dipole state $|n_r = 0, n_z = 1\rangle$. The plots of the three lowest odd modes (both $u_\alpha + v_\alpha$ and $u_\alpha - v_\alpha$) are shown in figures 11, 14 and 16.

An important feature of these functions is the fact that, due to interaction, coupling between the axial and the radial oscillations takes place even in the lowest excited mode, which in the non-interacting limit is purely radial ($|n_r = 1, n_z = 0\rangle$). The plots of the functions $u_1(r) + v_1(r)$ for different values of $N_0$ in figure 12 illustrate this feature because, as $N_0$ grows, the oscillation along the $z$-axis becomes more evident. This behaviour is confirmed by the experiment [3], where both radial and axial oscillations are observed even in the lowest normal mode. However, the function $u_1(r) - v_1(r)$ of figure 13 displays little or no coupling between the axial and radial oscillations. A stronger coupling is found as we consider higher frequencies, as shown in figures 15 and 16 that illustrate respectively the second even and the third odd normal mode.

Finally, we find a reasonable agreement between experimental data and our computed lowest eigenfrequency (fig. 12) as a function of the number of particles in the condensate $N_0$. This comparison is shown in figure 17.

\footnote{The functions $u_\alpha$ and $v_\alpha$ are both real because, as mentioned above, the ground-state vector of coefficients $a$ is real. Moreover it is useful to separate the oscillation $u_\alpha(r)e^{-i\omega_\alpha t} + v_\alpha(r)e^{i\omega_\alpha t}$ in real and imaginary part whose modulus are respectively $u_\alpha(r) + v_\alpha(r)$ and $u_\alpha(r) - v_\alpha(r)$. These distinct functions are both eigenfunctions of the Bogoliubov Hamiltonian of eq. 99 associated to the eigenfrequencies $\pm \omega_\alpha$.}
Figure 12: Functions $u(r) + v(r)$ describing the displacement in the lowest positive-$\omega_\alpha$ even normal mode for (top to bottom) $N_0 = 500$, $N_0 = 1500$, and $N_0 = 4500$. A significant admixture of radial and axial motion is evident.
Figure 13: Function $u(r) - v(r)$ relative to the lowest even normal mode with $N_0 = 4500$. Note that this oscillation is mostly radial, differently from its corresponding real part $u(r) + v(r)$ shown in figure 12.
Figure 14: Displacement functions $u(r) + v(r)$ and $u(r) - v(r)$ relative to the third odd normal mode, with $N_0 = 4500$. This mode represents a combination of a radial and axial excitation.
Figure 15: Displacement functions $u(r) + v(r)$ and $u(r) - v(r)$ relative to the second even normal mode, with $N_0 = 4500$. This mode represents an evident combination of a radial and axial excitation in the real part, while in the imaginary displacement part is purely radial.
Figure 16: Displacement functions $u(r) + v(r)$ and $u(r) - v(r)$ relative to the fourth odd normal mode, with $N_0 = 4500$. This mode represents a combination of a radial and axial excitation, especially in the real part.
Figure 17: Evolution of the frequency of the lowest even normal-mode with the number of condensate atoms. The triangles are experimental data from ref. [3] for $m = 0$ symmetry oscillation. The black curve represents our numerical calculation.
5 Discussion and Conclusion

In the present thesis we realize a simple implementation of the solution of GP and BdG equations for the mean-field ground state and excitation of a BEC. Although the method is not especially new \cite{5,8}, it is to be seen as a first building block of a project to compute the collective excitations of boson-fermion mixture, which are under consideration today. The expansion over the harmonic eigenstates provides a quite effective and numerically stable implementation of this method. Further developments of this work could be extending the same method to compute the ground-state and the excitation modes of a droplet of fermions, and next to solve the same problem for a mixture of interacting bosons and fermions. Moreover, it could be interesting to use one of these bosonic-fermionic normal-modes as the initial state of a Crank-Nicolson integration \cite{14,15} of the time-dependent GPE of eq. \cite{10} in order to investigate eventual non-linear coupling effects among the normal modes and how this non-linear coupling depends on the interaction strenght.
A The Gaussian Quadrature Method

The integrals of equations (47) and (48) are evaluated numerically by means of the Gaussian Quadrature Integration Method [10]. This method allows us to reduce these integrals to finite sums. It applies mainly to integrals of the form
\[ \int W(x)f(x)dx \]
where \( W(x) \) is a weight function like, for example, \( e^{-x^2} \) or \( e^{-x} \), which usually defines a scalar product on a function-space as:
\[ \langle f | g \rangle = \int W(x)f(x)g(x)dx. \]  

(76)

As we know, for a given \( W(x) \) and its corresponding functional scalar product, it is possible to determine an orthonormal basis \( \{ \varphi_j(x) \} \). The idea behind the Gaussian Quadrature Method consists in calculating the abscissa \( x_j \) and the corresponding weights \( w_j \) satisfying the following relation:
\[ \int_a^b W(x)f(x)dx \simeq \sum_{j=1}^{N_q} w_j f(x_j). \]  

(77)

By the fundamental theorem of Gaussian Quadrature, it turns out that the \( x_j \) with \( j = 1, ..., N_q \) are just the \( N_q \) roots of the \( N_q \)-th orthonormal eigenfunction \( \varphi_{N_q}(x) \) in the interval \((a, b)\). Moreover, if \( f(x) \) is a polynomial, which is the case of interest for us because our \( f(x) \) is a product of four Hermite and Laguerre polynomials, then the result is exact if \( N_q \) is large enough. Precisely, to obtain exact results in our case, the following two conditions must be satisfied:
\[ N_q^{(\Upsilon)} > 2 \max(n_r), \]
\[ N_q^{(\Xi)} > 2 \max(n_z). \]  

(78)
References


