STRUTTURA DELLA MATERIA 1

Problems on the equilibrium statistics of electromagnetic radiation

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The general context is quantum-mechanical electromagnetic fields at equilibrium.

1 The normal modes of the electromagnetic fields

Identify the normal modes of the electromagnetic fields in an empty cavity.

1.1 Solution

Inside an empty cavity, the electromagnetic fields follow the Maxwell equations in vacuum:

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\boldsymbol{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{2}$$

$$\boldsymbol{\nabla} \cdot \vec{E} = 0 \tag{3}$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0\,. \tag{4}$$

Take the rotation at both sides of Eq. (2):

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{E}) = -\boldsymbol{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$
$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \vec{B}\,.$$

Here, Eq. (3) eliminates the first term (containing $\boldsymbol{\nabla} \cdot \vec{E}$) at the left-hand side.
The right-hand side is conveniently substituted using Eq. (2):

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right].$$

Reorganizing:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \nabla^2 \vec{E} = \vec{0}$$
$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{E} = \vec{0} \tag{5}$$
$$\Box \vec{E} = \vec{0}\,.$$

The symbol $\Box$ indicates the d'Alembert operator. This equation indicates that in vacuum each component of the electric field independently satisfies a free wave equation. With simple manipulations one can check that also

$$\Box\vec{B} = \vec{0}\,.$$

Note the similarity of the differential operator in Eq. (5) with the Schrödinger operator

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2M}\nabla^2\right]\,.$$

For a wave equation, we search solutions of the type

$$\vec{E}(\vec{r},t) = \boldsymbol{\varepsilon} e^{i\vec{k}\cdot\vec{r}} q(t). \tag{6}$$

$\boldsymbol{\varepsilon}$ is a fixed constant vector (the polarization).

In principle the fields should satisfy appropriate boundary conditions at the cavity surface. For simplicity, like we did for the Schrödinger eigenfunctions, we just replace the actual cavity with a macroscopically large cubic box of volume $V = L \times L \times L$ where we apply *periodic boundary conditions*.

The fields are the same at opposite faces of the box. The allowed values of the $u = x, y, z$ wave-vector components

$$k_u = \frac{2\pi}{L}\,n_u\,, \qquad n_u = 0, \pm1, \pm2, \pm3, \dots \tag{7}$$

We substitute the ansatz (6) in the equation:

$$\begin{aligned}
\vec{0} &= \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\boldsymbol{\varepsilon} e^{i\vec{k}\cdot\vec{r}} q(t)\\
&= \boldsymbol{\varepsilon}\left[\frac{1}{c^2}q''(t)e^{i\vec{k}\cdot\vec{r}} - q(t)\nabla^2 e^{i\vec{k}\cdot\vec{r}}\right]\\
&= \boldsymbol{\varepsilon}\left[\frac{1}{c^2}q''(t) + |\vec{k}|^2 q(t)\right] e^{i\vec{k}\cdot\vec{r}}
\end{aligned}$$

As the constant vector $\boldsymbol{\varepsilon}$ and the plane wave are generally nonzero factors, the quantity that needs to remain constantly equal to 0 is the quantity in square brackets:

$$\begin{aligned}
\frac{1}{c^2}q''(t) + |\vec{k}|^2 q(t) = 0&\\
q''(t) &= -c^2|\vec{k}|^2 q(t)\\
q''(t) &= -\omega^2 q(t)
\end{aligned} \tag{8}$$

where we have introduced the angular frequency

$$\omega \equiv \omega(\vec{k}) \equiv c|\vec{k}|\,.$$

Equation (8) is the equation for a classical harmonic oscillator of frequency ω.

Its solutions are of the type

$$q(t) = q_0 e^{\pm i\omega t}.$$

q_0 is a complex number fixing the amplitude and initial phase of the harmonic oscillation.

Following the expression (6), with the oscillatory solution, the field oscillates everywhere at the same angular frequency: this satisfies the definition of a *normal mode of oscillation*.

We substitute the ansatz (6) in the Maxwell Eq. (3).

$$0 = \boldsymbol{\nabla} \cdot \vec{E} = \boldsymbol{\nabla} \cdot \left[\boldsymbol{\varepsilon} e^{i\vec{k}\cdot\vec{r}} q(t)\right] = q(t)\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla} e^{i\vec{k}\cdot\vec{r}} = q(t)\boldsymbol{\varepsilon} \cdot (i\vec{k}) e^{i\vec{k}\cdot\vec{r}}.$$

This equation implies

$$\boldsymbol{\varepsilon} \cdot \vec{k} = 0\,,$$

the *transverse-field condition*.

This condition implies 2 independent *polarizations* for each fixed $\vec{k}$. For example, if $\vec{k} = k\hat{z}$, then two independent polarizations are $\boldsymbol{\varepsilon} = \varepsilon\hat{x}$ or $\boldsymbol{\varepsilon} = \varepsilon\hat{y}$.

Each normal mode is identified by a $\vec{k}$ vector and a (perpendicular) polarization vector $\boldsymbol{\varepsilon}$.

The harmonic oscillator for each normal mode, follows a classical equation (8) for the variable $q = q_{\vec{k}\,\boldsymbol{\varepsilon}}$.

$q_{\vec{k}\,\boldsymbol{\varepsilon}}(t)$ provides the contribution of a normal mode to the total (classical) field amplitude at a given time t.

However in reality fields are quantum (as opposed to classical) objects. Maxwell's equations are the e.m.-fields equivalent to Newton's equations for the motion of particles.

The real quantum motion of the fields is described in detail by *quantum electrodynamics*.

Here we will consider a naive version of it, simply promoting the q classical variables to quantum operators.

Accordingly, to each normal mode we associate a quantum harmonic oscillator that satisfies a Schrödinger equation rather than the classical equation (8). We know everything about quantum harmonic oscillators, in particular their energy spectrum:

$$E_{\vec{k}\,\boldsymbol{\varepsilon}} = \hbar\omega_{\vec{k}\,\boldsymbol{\varepsilon}} \left(v_{\vec{k}\,\boldsymbol{\varepsilon}} + \frac{1}{2}\right), \qquad v_{\vec{k}\,\boldsymbol{\varepsilon}} = 0, 1, 2, \ldots$$

Usually in the present context the zero-point contribution is omitted, but one should be aware that when $\frac{1}{2}\hbar\omega_{\vec{k}\,\boldsymbol{\varepsilon}}$ is summed over all infinitely many $\vec{k}$ and $\boldsymbol{\varepsilon}$, it totals an infinite energy (and energy density too)!

2 The density of normal-mode frequencies of the electromagnetic fields

Determine the distribution of normal-mode frequencies or energies of the electromagnetic fields in an empty cavity.

2.1 Solution

From the previous exercise we know that normal modes are identified by a $\vec{k}$ vector and a (perpendicular) polarization vector $\boldsymbol{\varepsilon}$. Correspondingly, each mode has a frequency

$$\omega_{\vec{k}\,\boldsymbol{\varepsilon}} \equiv \omega(\vec{k}) \equiv c|\vec{k}| = c\frac{2\pi}{L}|\vec{n}|\,, \qquad n_u = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The frequency does not depend on polarization, and thanks to the transverse condition, each $\vec{k}$ has $g_s = 2$ independent oscillators with orthogonal polarizations.
The number of oscillators whose quantum is smaller or equal than a given energy $\mathcal{E}$ is

$$
\begin{aligned}
N(\hbar\omega \le \mathcal{E}) &= \sum_{\vec{n},\text{ such that } \hbar c \frac{2\pi}{L}|\vec{n}| \le \mathcal{E}} g_s \\
&= g_s \times \frac{4\pi}{3} \left(\frac{\mathcal{E}}{\hbar c \frac{2\pi}{L}} \right)^3 \\
&= g_s \times \frac{4\pi}{3} \times \frac{L^3 \mathcal{E}^3}{\hbar^3 c^3 8\pi^3} \\
&= \frac{g_s}{6\pi^2} \frac{V \mathcal{E}^3}{\hbar^3 c^3}
\end{aligned}
$$

The energy- density of these oscillator quanta is the derivative of this number of oscillators with respect to energy:

$$
\begin{aligned}
g_s g^{\text{ph}}(\mathcal{E}) &= \frac{dN(\hbar\omega \le \mathcal{E})}{d\mathcal{E}} \\
&= \frac{d}{d\mathcal{E}} \frac{g_s}{6\pi^2} \frac{V \mathcal{E}^3}{\hbar^3 c^3} \\
&= \frac{g_s}{2\pi^2} \frac{V \mathcal{E}^2}{\hbar^3 c^3} \\
&= \frac{V}{\pi^2 \hbar^3 c^3} \mathcal{E}^2
\end{aligned}
$$

[see Eq. (4.114) in Manini's book].
This density of oscillators allows us to replace

$$
\sum_{\text{normal modes}} \cdots \to \int_0^\infty d\mathcal{E}\, g_s g^{\text{ph}}(\mathcal{E}) \cdots
$$

3 The equilibrium statistics of the electromagnetic fields

Determine the thermodynamic functions, and the average number of photons for the electromagnetic fields at equilibrium in an empty cavity.

3.1 Solution

Recall the harmonic-oscillator statistics derived for the vibrational degree of freedom of molecules. Calling $x = \hbar\omega/(k_B T)$, here we copy the results obtained for that single-oscillator problem:

$$
Z_{1\,\text{vib}} = \frac{1}{2\sinh(x/2)}\,. \tag{9}
$$

$$
F_{1\,\text{vib}} = k_B T \ln(2\sinh(x/2))\,. \tag{10}
$$

$$
U_{1\,\text{vib}} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^x - 1}\,. \tag{11}
$$

$$[v] \equiv [n_{\mathcal{E}}]_{\mathrm{B}} = \frac{1}{e^x - 1}\,. \tag{12}$$

$$C_{V\,1\,\mathrm{vib}} = k_{\mathrm{B}} \left[\frac{x/2}{\sinh(x/2)}\right]^2. \tag{13}$$

$$S_{1\,\mathrm{vib}} = k_{\mathrm{B}} \frac{x}{e^x - 1} - k_{\mathrm{B}} \ln\left(1 - e^{-x}\right). \tag{14}$$

In our box we have many such harmonic oscillators. They are exactly non-interacting systems. Therefore, the total quantities can be obtained by summing over all these independent oscillators. One just needs to pay attention to the fact that each oscillator has its own frequency, and therefore its own dimensionless ratio

$$x \equiv x_{\vec{k}\,\boldsymbol{\varepsilon}} \equiv \frac{\hbar\omega_{\vec{k}\,\boldsymbol{\varepsilon}}}{k_{\mathrm{B}}T}\,.$$

$$\begin{aligned} U &= \sum_{\text{oscillators}} U_{1\,\mathrm{vib}}(\mathcal{E} = \hbar\omega_{\vec{k}\,\boldsymbol{\varepsilon}}) \\ &= \int_0^\infty d\mathcal{E}\, g_s g^{\mathrm{ph}}(\mathcal{E}) U_{1\,\mathrm{vib}}(\mathcal{E}) \\ &= \int_0^\infty d\mathcal{E}\, \frac{V}{\pi^2\hbar^3c^3}\, \mathcal{E}^2 \frac{\mathcal{E}}{e^{\beta\mathcal{E}} - 1} \\ &= V \int_0^\infty d\mathcal{E}\, \frac{1}{\pi^2\hbar^3c^3}\, \frac{\mathcal{E}^3}{e^{\beta\mathcal{E}} - 1} \\ &= V \int_0^\infty u(\mathcal{E}, T)\, d\mathcal{E}\,, \end{aligned} \tag{15}$$

where we have introduced

$$u(\mathcal{E}, T) = \frac{1}{V}\, g_s\, g_{\mathrm{ph}}(\mathcal{E})\, [n_{\mathcal{E}}]_{\mathrm{B}}\, \mathcal{E} = \frac{1}{\pi^2\hbar^3c^3}\, \frac{\mathcal{E}^3}{e^{\mathcal{E}/k_{\mathrm{B}}T} - 1}\,. \tag{16}$$

This quantity is the (double) density of internal energy per unit energy interval and per unit volume of the fields at equilibrium. Its units are J m^{-3} J^{-1} = m^{-3}.

It is possible to carry out the integration in Eq. (15):

$$\begin{aligned} U &= \frac{V}{\pi^2\hbar^3c^3} \int_0^\infty \frac{\mathcal{E}^3}{e^{\beta\mathcal{E}} - 1} d\mathcal{E} \\ &= \frac{V}{\pi^2\hbar^3c^3}\, (k_{\mathrm{B}}T)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx \\ &= \frac{V}{\pi^2\hbar^3c^3}\, (k_{\mathrm{B}}T)^4 \times \frac{\pi^4}{15} \\ &= V\, \frac{\pi^2(k_{\mathrm{B}}T)^4}{15\,\hbar^3c^3}\,. \end{aligned} \tag{17}$$

Likewise, the average number of photons:

$$\begin{aligned}
[\hat{N}] &= \int_0^\infty g_s\, g_{\rm ph}(\mathcal{E})[n_\mathcal{E}]_{\rm B}\, d\mathcal{E} \\
&= \int_0^\infty \frac{V}{\pi^2\hbar^3c^3}\,\mathcal{E}^2\frac{1}{e^{\beta\mathcal{E}}-1}\,d\mathcal{E} \\
&= \frac{V}{\pi^2\hbar^3c^3}\int_0^\infty \frac{\mathcal{E}^2}{e^{\beta\mathcal{E}}-1}\,d\mathcal{E} \\
&= \frac{V}{\pi^2\hbar^3c^3}(k_{\rm B}T)^3\int_0^\infty \frac{x^2}{e^x-1}dx \\
&= V\,\frac{2\,\xi(3)}{\pi^2\hbar^3c^3}\,(k_{\rm B}T)^3\,, \qquad (18)
\end{aligned}$$

where ξ is the Riemann function [$\xi(3) \simeq 1.20206$].
Similar expressions can be obtained for the total free energy F and entropy S, applying the same technique for summing over the oscillators.

4 The spectral irradiance of a blackbody cavity

Determine the spectral irradiance emitted by a blackbody cavity. Evaluate its total value for $T = 1$ K, 100 K, 300 K, 1000 K and 5000 K.

4.1 Solution

The total irradiance

$$R \equiv \frac{\text{total radiated power}}{\text{area of the hole}}$$

and its units are W m^{-2}.
Its spectral decomposition as a function of the photon energy $\mathcal{E} = \hbar\omega$ is

$$R = \int_0^\infty R(\mathcal{E})d\mathcal{E}$$

The units of $R(\mathcal{E})$ are W m^{-2} J^{-1} = m^{-2} s^{-1}.
To recall that we deal with a specific blackbody situation, we add a B label, plus the relevant cavity temperature:

$$R_{\rm B}(T)$$

and

$$R_{\rm B}(\mathcal{E}, T)$$

This last quantity takes contributions from all photons of the correct energy $\mathcal{E}$ exiting from the cavity, i.e. crossing an infinitesimal area dA in the outbound direction:

$$
\begin{aligned}
R_{\mathrm{B}}(\mathcal{E},T)d\mathcal{E}dA &= u(\mathcal{E},T)d\mathcal{E}dA \times \int_{\text{half sphere}} \frac{d\Omega}{4\pi} c\cos\theta \\
&= u(\mathcal{E},T)d\mathcal{E}dA\,\frac{c}{4\pi} \times \int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\theta\sin\theta d\theta \\
&= u(\mathcal{E},T)d\mathcal{E}dA\,\frac{c}{4\pi} \times 2\pi \int_0^{1} \cos\theta\, d\cos\theta \\
&= u(\mathcal{E},T)d\mathcal{E}dA\,\frac{c}{2} \times \int_0^{1} x\,dx \\
&= u(\mathcal{E},T)d\mathcal{E}dA\,\frac{c}{2} \times \left.\frac{x^2}{2}\right|_0^1 \\
&= \frac{c}{4} u(\mathcal{E},T)d\mathcal{E}dA
\end{aligned}
$$

This result gives us the searched for spectral irradiance:

$$
R_{\mathrm{B}}(\mathcal{E},T) = \frac{c}{4} u(\mathcal{E},T) = \frac{1}{4\pi^2\hbar^3c^2}\,\frac{\mathcal{E}^3}{e^{\mathcal{E}/k_{\mathrm{B}}T}-1}\,. \tag{19}
$$

By integrating over all energies, we obtain the total irradiance:

$$
\begin{aligned}
R_{\mathrm{B}}(T) &= \int_0^\infty R_{\mathrm{B}}(\mathcal{E},T)\,d\mathcal{E} = \int_0^\infty \frac{c}{4} u(\mathcal{E},T)\,d\mathcal{E} = \frac{c}{4}\int_0^\infty u(\mathcal{E},T)\,d\mathcal{E} \\
&= \frac{c}{4}\,\frac{\pi^2(k_{\mathrm{B}}T)^4}{15\,\hbar^3c^3} = \frac{\pi^2(k_{\mathrm{B}}T)^4}{60\,\hbar^3c^2}
\end{aligned} \tag{20}
$$

[see Eq. (4.118) in Manini's book].

The prefactor

$$
\sigma = \frac{\pi^2 k_{\mathrm{B}}^4}{60\,\hbar^3c^2} = 5.6704\times10^{-8}\,\frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}^4}
$$

multiplying T^4 to obtain the total irradiance is named Stefan-Boltzmann constant, and Eq. (20) is the Stefan-Boltzmann law, that can be also formulated as follows:

$$
R_{\mathrm{B}}(T) = \sigma\,T^4 = \frac{c}{4}\,\frac{U}{V}\,.
$$

Using the SB law, we obtain the requested values of irradiance:

T [K]	$R_{\mathrm{B}}(T)$ [W/m^2]
1	5.56×10^{-8}
100	5.56
300	459.3
1000	56704
5000	3.5×10^{7}

5 The solar panel "wasted" power

Under normal conditions, a thermal solar panel heats gets heated by solar radiation and transfers heat to a flux of water. At the same time, a solar panel behaves as a blackbody cavity, and emits blackbody radiation at its own temperature. This emitted radiation power is "wasted" because it detracts from the panel purpose.

On the other hand, if for any reason the water flux stops, the wasted power remains the only relevant heat sink, with the beneficial effect of limiting the panel's own temperature, thus preventing damage. What is this limiting temperature in the no-flux condition, assuming a perfectly isolated and perfectly absorbing panel facing solar radiation (solar constant $S_c = 1388\ \mathrm{W/m^2}$) at normal incidence?

5.1 Solution

The power emitted per unit surface by the panel at temperature T must be equated to the power it receives per unit surface, because this is the steady condition where the panel total energy remains constant:

$$\sigma T^4 = S_c \,.$$

Inverting this relation

$$T = \left(\frac{S_c}{\sigma}\right)^{1/4} = 395.5\ \mathrm{K} \simeq 122^\circ\mathrm{C}\,.$$

6 The heat capacity of "nothingness"

Determine the heat capacity per unit volume of vacuum as a function of temperature T, and evaluate it at $T = 1000$ K.

Determine the number density of a Ne gas that has the same heat capacity per unit volume as that of vacuum.

6.1 Solution

$$U = V\,\frac{\pi^2 (k_\mathrm{B} T)^4}{15\,\hbar^3 c^3}$$

Therefore

$$\begin{aligned}
C_{V\ \mathrm{vol}} &= \frac{\partial}{\partial T}\frac{U}{V} = \frac{\partial}{\partial T}\frac{\pi^2 (k_\mathrm{B} T)^4}{15\,\hbar^3 c^3} \\
&= \frac{\pi^2 k_\mathrm{B}^4}{15\,\hbar^3 c^3}\frac{\partial T^4}{\partial T} = \frac{\pi^2 k_\mathrm{B}^4}{15\,\hbar^3 c^3} 4T^3 \\
&= \frac{4\pi^2 k_\mathrm{B}^4 T^3}{15\,\hbar^3 c^3} \\
&= 3.02631 \times 10^{-15}\ \frac{\mathrm{J}}{\mathrm{m^3\,K^4}} \times T^3 \,.
\end{aligned}$$

$$C_{V\ \mathrm{vol}}(T = 1000\ \mathrm{K}) = 3.02631 \times 10^{-6}\ \frac{\mathrm{J}}{\mathrm{m^3\,K}}\,.$$

The internal energy of a monoatomic gas at such high temperatures is

$$U_\mathrm{tr} = \frac{3}{2} N k_\mathrm{B} T \,.$$

As a consequence, the heat capacity per unit volume is

$$C_{V\ \mathrm{tr\ vol}} = \frac{1}{V}\frac{\partial U_\mathrm{tr}}{\partial T} = \frac{3}{2}\frac{N}{V} k_\mathrm{B} \,.$$

We must equate this quantity to the homologous quantity for the photons, and obtain the density N/V:

$$C_{V\ \mathrm{vol}} = C_{V\,\mathrm{tr\ vol}}$$

$$\frac{4\pi^2 k_\mathrm{B}^4 T^3}{15\,\hbar^3 c^3} = \frac{3}{2}\frac{N}{V}k_\mathrm{B}$$

$$\frac{N}{V} = \frac{2}{3}\frac{1}{k_\mathrm{B}}\frac{4\pi^2 k_\mathrm{B}^4 T^3}{15\,\hbar^3 c^3} = \frac{8\pi^2 (k_\mathrm{B}T)^3}{45\,\hbar^3 c^3} = \frac{8\pi^2}{45}\left(\frac{k_\mathrm{B}T}{\hbar c}\right)^3$$

Numerically:

$$\frac{N}{V} = 1.7546 \times \left(\frac{1.3806 \times 10^{-20}\ \mathrm{J}}{3.161536 \times 10^{-26}\ \mathrm{J\ m}}\right)^3 = 1.461 \times 10^{17}\ \mathrm{m}^{-3}.$$

Would we call this a high or a low density? The pressure of such a gas is

$$P = \frac{N}{V}k_\mathrm{B}T = 0.0020\ \mathrm{Pa}\,.$$

In lab practice, this is what one calls *high vacuum* (10^{-1}–10^{-7} Pa)!

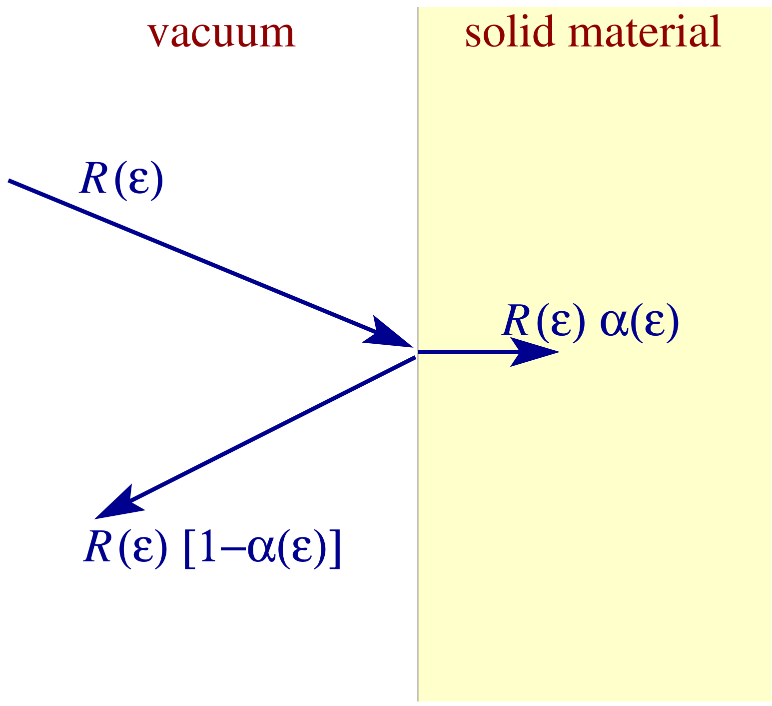
7 The "blackbody" name

Explain what the equilibrium radiation in a cavity has to do with black bodies.

7.1 Solution

We need to first of all clarify what "black" means.

We start inspecting the following figure:



The coefficient $\alpha(\mathcal{E})$ is the absorption coefficient of the material. $\alpha(\mathcal{E})$ represents the percentage of the radiated power that a opaque surface absorbs (and transforms into internal energy – heat).

As the figure illustrates, due the energy conservation, what is not absorbed is reflected (if the surface is optically smooth) or diffused (as occurs for more standard non-polished surfaces): $[1 - \alpha(\mathcal{E})]$ is the re-emitted percentage.

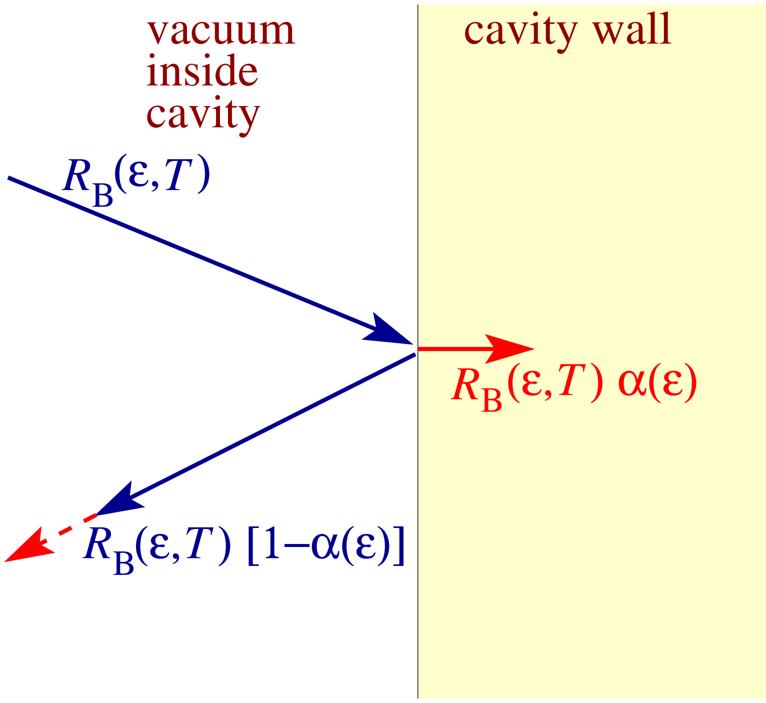
$\alpha(\mathcal{E})$ depends on the photon energy $\mathcal{E} = \hbar\omega$ (or equivalently frequency $\nu = \omega/(2\pi)$, or wavelength $\lambda = c/\nu$), indicating that radiation of different "colors" can be absorbed in different percentages by a surface. For example, a "red" object has a small $\alpha(\mathcal{E})$ in the red region, near 1.8 eV, and a large $\alpha(\mathcal{E})$ elsewhere (at least in the visible range, up to $\sim$ 4 eV – our eyes are not sensitive to the infrared and ultraviolet portions of the spectrum, which means that the value of $\alpha(\mathcal{E})$ there does not affect an object's perceived color). As a result, $[1 - \alpha(\mathcal{E})]$ peaks in the red region, and when illuminated with white light, the object gives back mostly red light, while absorbing the rest of the spectrum.

A white object has $\alpha(\mathcal{E}) \equiv 0$.

A gray object has $\alpha(\mathcal{E}) \equiv$ a constant (photon-energy-independent) intermediate value, e.g. 0.2 for light gray, 0.7 for dark gray.

A black object has $\alpha(\mathcal{E}) \equiv 1$.

Now that we know the quantitative meaning of colored and black surfaces, consider the isothermal cavity inner surface:



Every spot of this inner surface is of course hit by the equilibrium blackbody radiation, the one whose irradiance $R_B(\mathcal{E}, T)$ is described by Eq. (19).

Given the surface chromatic properties summarized by its spectral absorption coefficient $\alpha(\mathcal{E})$, a fraction $[1 - \alpha(\mathcal{E})]$ of this incoming radiation flux gets reflected/diffused by the surface back into the cavity.

However, this reflected power cannot be all of it: if it was, energy would be drained away from the fields inside cavity volume into the walls.

For this reason, the energy of the fields inside the cavity must be replenished by the cavity wall itself.

The missing power (per unit surface and unit spectral interval) is precisely the absorbed part:

$$R_B(\mathcal{E},T) - R_B(\mathcal{E},T)[1-\alpha(\mathcal{E})] = R_B(\mathcal{E},T)\alpha(\mathcal{E})\,.$$

This radiance (dashed arrow in figure) is what the cavity wall *must emit* to guarantee the energy balance at equilibrium.

But this means that the cavity wall emits this $R_B(\mathcal{E},T)\alpha(\mathcal{E})$ spectrum "spontaneously", regardless of it being part of the cavity, just because it is kept at a temperature T.

We could take the cavity apart, and that piece of cavity wall would still emit spontaneously the spectrum $R_B(\mathcal{E},T)\alpha(\mathcal{E})$.

The cavity wall is nothing but a generic piece of opaque material, which may have never had anything to do with any cavity. The argument above guarantees that *any object kept at a temperature T radiates e.m. radiation whose spectral irradiance is $R_B(\mathcal{E},T)\alpha(\mathcal{E})$.*

We have now all elements to solve the problem: a black body has $\alpha(\mathcal{E}) \equiv 1$, and therefore it emits a spectrum

$$R_B(\mathcal{E},T)\alpha(\mathcal{E}) = R_B(\mathcal{E},T)\times 1 = R_B(\mathcal{E},T)\,.$$

The spectrum emitted by a perfectly black body is identical to the equilibrium spectrum of a cavity kept at equilibrium at the same temperature as that of the surface of the black body.

This justifies and clarifies the name *blackbody radiation*.

8 Planetary radiative balance

The Sun's radiation strikes the Earth surface with a total $G_{SC} = 1361$ W/m^2 mean power density (called solar constant).

Under the simplifying assumption that the Sun's and the Earth's surfaces are both approximately perfectly black ($\alpha \equiv 1$), estimate:

1. the temperature of the Sun's surface, given its radius $r_{\text{Sun}} = 6.95\times 10^8$ m, and the average Sun-Earth distance $d = 1.49\times 10^{11}$ m;
2. the average Earth temperature, assuming that its heat conductivity is infinitely large, and that a steady state has been reached in the past.

8.1 Solution

1. The total power radiated by the Sun equals $S_{\text{Sun}}\sigma T^4_{\text{Sun}}$.

   Neglecting any opacity of the space in between the Sun and the Earth's orbit, all that power reaches and crosses all spheres centered around the Sun, including the one whose radius coincides with d (and whose area is therefore $4\pi d^2$).

   This power is $G_{SC}\times 4\pi d^2$.

   Equating these two quantities we can obtain the temperature of the Sun's surface:

   $$4\pi r^2_{\text{Sun}}\sigma T^4_{\text{Sun}} = G_{SC}\times 4\pi d^2,$$

   $$T^4_{\text{Sun}} = \frac{G_{SC} d^2}{r^2_{\text{Sun}}\sigma},$$

$$
\begin{aligned}
T_{\text{Sun}} &= \left(\frac{G_{SC}}{\sigma}\right)^{1/4}\left(\frac{d}{r_{\text{Sun}}}\right)^{1/2} \\
&= \left(\frac{1361\ \text{W/m}^2}{5.6704\times 10^{-8}\ \frac{\text{W}}{\text{m}^2\,\text{K}^4}}\right)^{1/4}\left(\frac{1.49\times 10^{11}\ \text{m}}{6.95\times 10^{8}\ \text{m}}\right)^{1/2} \\
&= \left(2.400\times 10^{10}\ \text{K}^4\right)^{1/4}(214.4)^{1/2} \\
&= 5763.18\ \text{K}\,.
\end{aligned}
$$

2. As $d \gg r_{\text{Sun}} \gg r_{\text{Earth}}$, the solar radiation hits the Earth in a way that closely resembles a parallel beam.

   This radiation hits the "day" side of the Earth.

   This is 50% of the Earth surface, and it is hit at varying angles.

   However, the actual perpendicular cross-sectional surface of the planet equals the area of a circle with the same radius:

$$
\pi r_{\text{Earth}}^2\,.
$$

   In the black-surface (fully absorbing, null *albedo*) hypothesis, the total instantly-received power equals

$$
P_{\text{in}} = G_{SC}\times \pi r_{\text{Earth}}^2\,.
$$

   On the other hand, we know that a black surface radiates a blackbody spectrum, at its temperature T_{Earth} (assumed constant and stationary). The total emitted power follows the Stefan-Boltzmann law:

$$
P_{\text{out}} = \sigma T_{\text{Earth}}^4 \times 4\pi r_{\text{Earth}}^2\,,
$$

   where the entire sphere surface has to be taken into account.

   The stationary condition (neglecting all other power sources) requires that

$$
P_{\text{in}} = P_{\text{out}}\,.
$$

   This is solved for T_{Earth}:

$$
\begin{aligned}
G_{SC}\times \pi r_{\text{Earth}}^2 &= \sigma T_{\text{Earth}}^4 \times 4\pi r_{\text{Earth}}^2\,, \\
G_{SC} &= 4\,\sigma T_{\text{Earth}}^4\,, \\
T_{\text{Earth}}^4 &= \frac{G_{SC}}{4\,\sigma}\,,
\end{aligned}
$$

$$
T_{\text{Earth}} = \left(\frac{G_{SC}}{4\,\sigma}\right)^{1/4} = \left(\frac{1361\ \text{W/m}^2}{4\times 5.6704\times 10^{-8}\ \frac{\text{W}}{\text{m}^2\,\text{K}^4}}\right)^{1/4} = 278.32\ \text{K}\,.
$$

9 Written test 08/04/2014, problem 2

Si determini la massa a riposo persa dal sole ogni secondo per emissione di radiazione. Si assuma che la temperatura superficiale del sole sia 5700 K e il diametro 1.4×10^9 m.

9.1 Solution

Using the Stefan-Boltzmann law

$$P_{\rm Sun} = \sigma T_{\rm Sun}^4 \times 4\pi\, R_{\rm Sun}^2 = 3.6857 \times 10^{26}\ \mathrm{W}\,.$$

Using the Einstein's energy-mass relation

$$\frac{dM_{\rm Sun}}{dt} = -\frac{P_{\rm Sun}}{c^2} = -4.1 \times 10^9\ \frac{\mathrm{kg}}{\mathrm{s}}\,.$$

10 Peak emission

Evaluate the maximum-irradiance photon energy for blackbody radiation at a given temperature T.
Provide the value for $T = 300$ K and $T = 1000$ K.

10.1 Solution

$R_{\rm B}(\mathcal{E}, T) = \frac{c}{4} u(\mathcal{E}, T)$, so we can as well evaluate the maximum of

$$u(\mathcal{E}, T) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{\mathcal{E}^3}{e^{\beta\mathcal{E}} - 1} = \frac{1}{\pi^2 \hbar^3 c^3 \beta^3} \frac{\beta^3\mathcal{E}^3}{e^{\beta\mathcal{E}} - 1} = \frac{1}{\pi^2 \hbar^3 c^3 \beta^3} \frac{x^3}{e^{x} - 1}\,.$$

As $x = \beta\epsilon$, we will obtain the maximum of this universal (i.e. T-independent) function of x, say $x_{\rm max}$, and then the photon energy of this maximum is $\epsilon_{\rm max} = x_{\rm max}/\beta = x_{\rm max} k_{\rm B} T$. This observation proves Wien's displacement law, namely that $\epsilon_{\rm max}$ is proportional to the blackbody source temperature.
We compute $x_{\rm max}$ by taking the derivative of the universal function in the fraction with respect to x, and equating it to zero:

$$\frac{d}{dx} \frac{x^3}{e^x - 1} = \frac{3x^2(e^x - 1) - x^3\, e^x}{(e^x - 1)^2} = 0\,.$$

Simplify:

$$\begin{aligned} 3(e^x - 1) - x\, e^x &= 0 \\ 3e^{\beta\mathcal{E}} - 3 - x\, e^x &= 0 \\ (3 - x)\, e^x &= 3\,. \end{aligned}$$

Solve graphically: solutions for $x = 0$ (trivial) and for $x = x_{\rm max} \simeq 2.82144$.
In conclusion:

$$\mathcal{E}_{\rm max}(T) = 2.82144\, k_{\rm B} T \qquad \text{[explicit form of Wien's law]}\,.$$

Examples:

$$\begin{aligned} \mathcal{E}_{\rm max}(300\ \mathrm{K}) &= 1.1686 \times 10^{-20}\ \mathrm{J} = 0.0729\ \mathrm{eV} \\ \mathcal{E}_{\rm max}(1000\ \mathrm{K}) &= 3.8954 \times 10^{-20}\ \mathrm{J} = 0.2431\ \mathrm{eV}\,. \end{aligned}$$

11 Peak emission (wavelength version)

Same as previous exercise, but as a function of wavelength.

11.1 Solution

$$\bar{u}(\lambda, T) = \frac{16\pi^2\,\hbar\,c\,\lambda^{-5}}{\{\exp[2\pi\hbar c/(\lambda k_\mathrm{B}T)] - 1\}}.$$

Proceed as above, differentiating with respect to λ.
Result:

$$\lambda_\mathrm{max}(T) = 0.201405\frac{hc}{k_\mathrm{B}T}\,.$$

Note that

$$\mathcal{E}_\mathrm{max}(T) \neq \frac{hc}{\lambda_\mathrm{max}(T)}\,!$$

12 Photon counting

How many photons per cubic meter have energy $\hbar\omega$ in between $\mathcal{E}_\mathrm{max}(T)$ and $1.05\,\mathcal{E}_\mathrm{max}(T)$ in a blackbody radiation field at 300 K?

12.1 Solution

$$[\hat{N}] = \frac{V}{\pi^2\hbar^3c^3}\int_0^\infty \frac{\mathcal{E}^2}{e^{\beta\mathcal{E}}-1}\,d\mathcal{E}$$

is the total number of photons see Eq. (18).
The required number is instead

$$\begin{aligned}\frac{1}{V}\,[\hat{N}](\mathcal{E}_\mathrm{max}(T) \le \mathcal{E} \le 1.05\,\mathcal{E}_\mathrm{max}(T)) &= \frac{1}{\pi^2\hbar^3c^3}\int_{\mathcal{E}_\mathrm{max}}^{1.05\,\mathcal{E}_\mathrm{max}} \frac{\mathcal{E}^2}{e^{\beta\mathcal{E}}-1}\,d\mathcal{E} \\ &= \frac{(k_\mathrm{B}T)^3}{\pi^2\hbar^3c^3}\int_{x_\mathrm{max}}^{1.05\,x_\mathrm{max}} \frac{x^2}{e^x-1}\,dx\end{aligned}$$

Call $f(x) = x^2/(e^x - 1)$.
Using the trapezoidal integration

$$\begin{aligned}\frac{[\hat{N}](\mathcal{E}_\mathrm{max}(T) \le \mathcal{E} \le 1.05\,\mathcal{E}_\mathrm{max}(T))}{V} &\simeq \frac{(k_\mathrm{B}T)^3}{\pi^2\hbar^3c^3} \times \frac{1}{2} \times 0.05x_\mathrm{max}[f(x_\mathrm{max}) + f(x_\mathrm{max}\,1.05)] \\ &= \frac{(k_\mathrm{B}T)^3}{\pi^2\hbar^3c^3} \times 0.06928 \\ &= 1.58 \times 10^{13}\ \mathrm{m}^{-3}\end{aligned}$$

(Same value to all digits obtained by more refined integration.)
By the way, the total number, obtained integrating from 0 to ∞ instead, is

$$\frac{[\hat{N}]_\mathrm{tot}}{V} = 5.477 \times 10^{14}\ \mathrm{m}^{-3},$$

using Eq. (18).

13 Temperature measurements

Find the temperature of a cavity having a radiant energy density at $\lambda_1 = 200$ nm that is 3.82 times that at $\lambda_2 = 400$ nm.

13.1 Solution

$$\bar{R}(\lambda, T) = \frac{c}{4} u(\lambda, T)\,,$$

with

$$\bar{u}(\lambda, T) = \frac{16\pi^2\,\hbar\,c\,\lambda^{-5}}{\{\exp[(2\pi\hbar c)/(\lambda k_{\rm B} T)] - 1\}}.$$

Observe that $\lambda_2 = 2\lambda_1$.
We have

$$\begin{aligned}
\frac{\bar{R}(\lambda_1, T)}{\bar{R}(\lambda_2, T)} = \frac{\bar{R}(\lambda_1, T)}{\bar{R}(2\lambda_1, T)} &= \frac{\bar{u}(\lambda_1, T)}{\bar{u}(2\lambda_1, T)} \\
&= 2^5\,\frac{\exp(hc\beta/(2\lambda_1)) - 1}{\exp(hc\beta/\lambda_1) - 1} \\
&= 3.82
\end{aligned}$$

call $y = \exp(hc\beta/(2\lambda_1))$:

$$2^5\,\frac{y-1}{y^2-1} = 3.82\,,$$

$$32\,\frac{y-1}{(y-1)(y+1)} = 3.82\,,$$

$$32\,\frac{1}{y+1} = 3.82\,,$$

$$y + 1 = \frac{32}{3.82}\,,$$

$$y = \frac{32}{3.82} - 1 = 7.37696\,,$$

$$\frac{hc\beta}{2\lambda_1} = \ln y\,,$$

$$k_{\rm B} T = \frac{hc}{2\lambda_1 \ln y}\,,$$

$$T = \frac{hc}{2k_{\rm B}\lambda_1 \ln y} = 17999.3\ \text{K}\,.$$

14 Written test 16/11/2018, problem 3

Si assimili un corpo umano a un corpo grigio (riflettività pari al 30% su tutto l'intervallo spettrale rilevante) alla temperatura di 37°C. Si valutino: (i) la potenza totale che esso irraggia, assumendo una superficie totale di 1.7 m^2, e (ii) il numero di fotoni di lunghezza d'onda compresa tra 5.99 μm e 6.01 μm emessi per unità di tempo. Valutando il bilancio energetico radiativo tenendo conto anche della radiazione che lo raggiunge, si determini la potenza netta emessa dal corpo umano se immesso in una cavità di corpo nero all'equilibrio a $T = 30$°C. [Si ricorda l'espressione per la costante di Stefan-Boltzmann: $\pi^2 k_{\rm B}^4/(60\,\hbar^3 c^2)$.]

14.1 Solution

1. For the irradiated power, we apply the Stefan-Boltzmann law, with the appropriate $\alpha = 1 - 30\% = 0.7$ correction:

$$P_{\text{out}} = S \times \alpha \times \sigma T_{\text{body}}^4 = 1.7 \text{ m}^2 \times 0.7 \times 5.6704 \times 10^{-8} \frac{\text{W}}{\text{m}^2\,\text{K}^4} \times [(273+37) \text{ K}]^4$$
$$= 623.17 \text{ W}.$$

2. For a blackbody spectrum, the spectral number of photons per unit time and surface area is given by the ratio of the spectral irradiance to the photon energy:

$$\frac{d\Phi_{\text{phot}}}{dt}(\mathcal{E}) = \frac{R_{\text{B}}(\mathcal{E}, T)}{\mathcal{E}} = \frac{1}{\mathcal{E}} \frac{1}{4\pi^2\hbar^3c^2} \frac{\mathcal{E}^3}{e^{\mathcal{E}/k_{\text{B}}T} - 1} = \frac{1}{4\pi^2\hbar^3c^2} \frac{\mathcal{E}^2}{e^{\mathcal{E}/k_{\text{B}}T} - 1}.$$

The relevant energy range is

$$\mathcal{E}_{\text{min}} = \frac{hc}{6.01 \ \mu\text{m}} = 3.30523 \times 10^{-20} \text{ J}$$
$$\mathcal{E}_{\text{max}} = \frac{hc}{5.99 \ \mu\text{m}} = 3.31627 \times 10^{-20} \text{ J}.$$

We integrate the photon spectral flux in this range:

$$\begin{aligned}
\frac{d\Phi_{\text{phot}}}{dt}(\mathcal{E}_{\text{min}} < \mathcal{E} < \mathcal{E}_{\text{max}}) &= \int_{\mathcal{E}_{\text{min}}}^{\mathcal{E}_{\text{max}}} \frac{d\Phi_{\text{phot}}}{dt}(\mathcal{E}) d\mathcal{E} \\
&= \int_{\mathcal{E}_{\text{min}}}^{\mathcal{E}_{\text{max}}} \frac{1}{4\pi^2\hbar^3c^2} \frac{\mathcal{E}^2}{e^{\mathcal{E}/k_{\text{B}}T} - 1} d\mathcal{E} \\
&= \frac{1}{4\pi^2\hbar^3c^2} \int_{\mathcal{E}_{\text{min}}}^{\mathcal{E}_{\text{max}}} \frac{\mathcal{E}^2}{e^{\mathcal{E}/k_{\text{B}}T} - 1} d\mathcal{E} \\
&= \frac{1}{4\pi^2\hbar^3c^2} (k_{\text{B}}T)^3 \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{x^2}{e^x - 1} dx,
\end{aligned}$$

with the usual substitution $x = \mathcal{E}/(k_{\text{B}}T)$.

$x_{\text{min}} = 7.72249$; $x_{\text{max}} = 7.74828$.

We evaluate the dimensionless integral with the usual trapezoidal method:

$$\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{x^2}{e^x - 1} dx \simeq \frac{1}{2} \left[\frac{x_{\text{min}}^2}{e^{x_{\text{min}}} - 1} + \frac{x_{\text{max}}^2}{e^{x_{\text{max}}} - 1} \right] (x_{\text{max}} - x_{\text{min}}) = 0.000674681.$$

Substituting in the expression above:

$$\begin{aligned}
\frac{d\Phi_{\text{phot}}}{dt}(\mathcal{E}_{\text{min}} < \mathcal{E} < \mathcal{E}_{\text{max}}) &= \frac{1}{4\pi^2\hbar^3c^2} (k_{\text{B}}T)^3 \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{x^2}{e^x - 1} dx \\
&= 2.40309 \times 10^{83} \text{ J}^{-3}\,\text{s}^{-1}\,\text{m}^{-2} \times \\
&\quad \times 7.84033 \times 10^{-62} \text{ J}^3 \times 0.000674681 \\
&= 1.27117 \times 10^{19} \text{ s}^{-1}\,\text{m}^{-2}.
\end{aligned}$$

By multiplying the relevant surface area, we obtain

$$\begin{aligned}
\frac{dN_{\text{phot}}}{dt}(\mathcal{E}_{\text{min}} < \mathcal{E} < \mathcal{E}_{\text{max}}) &= S \times \frac{d\Phi_{\text{phot}}}{dt}(\mathcal{E}_{\text{min}} < \mathcal{E} < \mathcal{E}_{\text{max}}) \\
&= 1.7 \text{ m}^2 \times 1.27117 \times 10^{19} \text{ s}^{-1}\,\text{m}^{-2} \\
&= 2.16100 \times 10^{19} \text{ s}^{-1}.
\end{aligned}$$

This photon rate would be relevant for a properly black surface.

Here the surface has $\alpha = 0.7$, therefore:

$$\begin{aligned}\frac{dN_{\text{phot }[\alpha]}}{dt}(\mathcal{E}_{\min} < \mathcal{E} < \mathcal{E}_{\max}) &= \alpha \times \frac{dN_{\text{phot}}}{dt}(\mathcal{E}_{\min} < \mathcal{E} < \mathcal{E}_{\max}) \\ &= 0.7 \times 2.16100 \times 10^{19}\ \text{s}^{-1} \\ &= 1.51269 \times 10^{19}\ \text{s}^{-1}\,.\end{aligned}$$

3. Taking the surface area and the fractional absorbance α into account, the radiative balance of the human body is as follows:

$$\begin{aligned}P_{\text{tot}} &= P_{\text{out}} - P_{\text{in}} \\ &= S \times \alpha \times \sigma \left(T^4_{\text{body}} - T^4_{\text{cavity}}\right) \\ &= 1.7\ \text{m}^2 \times 0.7 \times 5.6704 \times 10^{-8}\, \frac{\text{W}}{\text{m}^2\,\text{K}^4} \left([(273+37)\ \text{K}]^4 - [(273+30)\ \text{K}]^4\right) \\ &= 54.408\ \text{W}\,.\end{aligned}$$

15 Written test 19/09/2019, problem 4

Si stimi la temperatura minima necessaria a produrre raggi ultravioletti di energia maggiore di 5 eV mediante una sorgente termica di corpo nero, se si desidera un'efficienza dello 0.1%. [Si ricorda che $\int_0^\infty dt\, t^3/(e^t - 1) = \pi^4/15$, e che per $t \gg 1$, rimpiazzando $[\exp(t) - 1]^{-1}$ con $\exp(-t)$ l'errore introdotto risulta del tutto trascurabile].

15.1 Solution

Consider the spectral irradiance

$$R(\mathcal{E}, T) = \frac{c}{4} u(\mathcal{E}, T) = \frac{1}{4\pi^2\hbar^3 c^2}\, \frac{\mathcal{E}^3}{e^{\mathcal{E}/k_\text{B}T} - 1}\,.$$

By definition, the efficiency η of a blackbody source for the $\mathcal{E} > \mathcal{E}_{\min} = 5$ eV spectral region is

$$\eta = \frac{\int_{\mathcal{E}_{\min}}^\infty R(\mathcal{E}, T) d\mathcal{E}}{\int_0^\infty R(\mathcal{E}, T) d\mathcal{E}}\,.$$

We can simplify many constants, and reformulate as dimensionless integration:

$$\begin{aligned}\eta &= \frac{\int_{\mathcal{E}_{\min}}^\infty \frac{\mathcal{E}^3}{e^{\mathcal{E}/k_\text{B}T} - 1} d\mathcal{E}}{\int_0^\infty \frac{\mathcal{E}^3}{e^{\mathcal{E}/k_\text{B}T} - 1} d\mathcal{E}} = \frac{(k_\text{B}T)^4 \int_{x_{\min}}^\infty \frac{x^3}{e^x - 1} dx}{(k_\text{B}T)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx} \\ &= \frac{\int_{x_{\min}}^\infty \frac{x^3}{e^x - 1} dx}{\int_0^\infty \frac{x^3}{e^x - 1} dx}\,,\end{aligned}$$

where $x_{\min} = \mathcal{E}_{\min}/(k_\text{B}T)$ is still to be determined.

So far everything is exact. We follow the suggestion, assuming $x \ge x_{\min} \gg 3$, so that $e^x \gg e^3 \simeq 20 \gg 1$. In the end, we should check if this is the case, once $x_{\min}$, and therefore T, is determined.

$$\eta \simeq \frac{\int_{x_{\rm min}}^{\infty} \frac{x^3}{e^x} dx}{\int_0^{\infty} \frac{x^3}{e^x - 1} dx} = \frac{\int_{x_{\rm min}}^{\infty} x^3\, e^{-x} dx}{\frac{\pi^4}{15}} \,,$$

The integral at the numerator can be evaluated by parts (3 times):

$$\begin{aligned}
\int_{x_{\rm min}}^{\infty} x^3\, e^{-x} dx &= x^3 \times (-e^{-x})\big|_{x_{\rm min}}^{\infty} - \int_{x_{\rm min}}^{\infty} 3x^2\, (-e^{-x}) dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3 \int_{x_{\rm min}}^{\infty} x^2\, e^{-x} dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3\, x^2 \times (-e^{-x})\big|_{x_{\rm min}}^{\infty} - 3 \int_{x_{\rm min}}^{\infty} 2x\, (-e^{-x}) dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3x_{\rm min}^2 e^{-x_{\rm min}} + 6 \int_{x_{\rm min}}^{\infty} x\, e^{-x} dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3x_{\rm min}^2 e^{-x_{\rm min}} + 6\, x \times (-e^{-x})\big|_{x_{\rm min}}^{\infty} - 6 \int_{x_{\rm min}}^{\infty} (-e^{-x}) dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3x_{\rm min}^2 e^{-x_{\rm min}} + 6x_{\rm min} e^{-x_{\rm min}} + 6 \int_{x_{\rm min}}^{\infty} e^{-x} dx \\
&= x_{\rm min}^3 e^{-x_{\rm min}} + 3x_{\rm min}^2 e^{-x_{\rm min}} + 6x_{\rm min} e^{-x_{\rm min}} + 6\, (-e^{-x})\big|_{x_{\rm min}}^{\infty} \\
&= \left(x_{\rm min}^3 + 3x_{\rm min}^2 + 6x_{\rm min} + 6\right) e^{-x_{\rm min}}
\end{aligned}$$

Plugging this result in the formula for η:

$$\eta \simeq \frac{\left(x_{\rm min}^3 + 3x_{\rm min}^2 + 6x_{\rm min} + 6\right) e^{-x_{\rm min}}}{\frac{\pi^4}{15}} = 0.1\%$$

$$f(x_{\rm min}) := \left(x_{\rm min}^3 + 3x_{\rm min}^2 + 6x_{\rm min} + 6\right) e^{-x_{\rm min}} = 0.001 \frac{\pi^4}{15} = 0.00649394 =: \bar{f} \,. \qquad (21)$$

This equation should allow us to determine $x_{\rm min}$ and thus the required minimum temperature.

Eq. (21) is a transcendental equation, which we solve numerically.

The function $f(x_{\rm min})$ starts off at 6 for $x_{\rm min} = 0$ and decays to 0 for large $x_{\rm min}$.

We have $f(5) = 1.590$, $f(10) = 0.0620 > \bar{f}$, $f(20) = 1.92 \times 10^{-5} < \bar{f}$, therefore the root is in the interval $10 < x_{\rm min} < 20$. We can use bisection to determine the root to arbitrary accuracy. For example,

$$x_{\rm min} \simeq 12.9619$$

is accurate to 6 significant digits. We see that $x_{\rm min}$ turns out much larger than 3, so we are satisfied with the approximation adopted earlier on.

We obtain a temperature

$$T = \frac{1}{x_{\rm min}} \frac{\mathcal{E}_{\rm min}}{k_{\rm B}} = 0.0771493 \times 58022.6 \text{ K} = 4476.4 \text{ K} \,.$$

16 Spontaneous/stimulated Emission

A gas of atomic H stands in equilibrium with the surrounding blackbody radiation field. Determine:

1. the ratio between the rates of spontaneous and stimulated emission at temperature $T = 300$ K for radiation associated to the transition 1s–2p;
2. the temperature which would make these two rates equal.

16.1 Solution

1. Recall

$$R_{1\to 2} = B_{12}\,\rho(\epsilon)\,, \tag{22}$$
$$R_{2\to 1} = A_{21} + B_{21}\,\rho(\epsilon)\,, \tag{23}$$

where "1" stands for 1s and "2" stands for any of the 2p components with $m_l = -1, 0, 1$.

The excitation energy for hydrogen is

$$\begin{aligned}\epsilon &= -\frac{1}{2}E_{\text{Ha}} \times \frac{1}{2^2} - \left(-\frac{1}{2}E_{\text{Ha}} \times \frac{1}{1^2}\right) = \frac{1}{2}E_{\text{Ha}}\left(1 - \frac{1}{4}\right) \\ &= \frac{3}{8}E_{\text{Ha}} = 1.6349 \times 10^{-18}\ \text{J} = 10.204\ \text{eV}\end{aligned}$$

(nonrelativistic approximation and reduced mass $\mu \simeq m_e$).

In the conditions of the present problem, the radiation spectral density is that of blackbody radiation:

$$\rho(\epsilon) = u(\epsilon, T) = \frac{1}{\pi^2\hbar^3c^3}\,\frac{\epsilon^3}{e^{\epsilon/k_\text{B}T} - 1}\,.$$

The requested ratio between the rates of spontaneous and stimulated emission can be computed by taking advantage of the Einstein equations

$$B_{12} = B_{21} = \frac{\pi^2\hbar^3c^3}{\epsilon^3}\,A_{21}\,, \tag{24}$$

as follows:

$$\begin{aligned}\frac{A_{21}}{B_{21}u(\epsilon,T)} &= \frac{A_{21}}{\dfrac{\pi^2\hbar^3c^3}{\epsilon^3}\,A_{21}\,u(\epsilon,T)} \\ &= \frac{1}{\dfrac{\pi^2\hbar^3c^3}{\epsilon^3}\,u(\epsilon,T)} \\ &= \frac{1}{\dfrac{\pi^2\hbar^3c^3}{\epsilon^3}\,\dfrac{1}{\pi^2\hbar^3c^3}\,\dfrac{\epsilon^3}{e^{\epsilon/k_\text{B}T} - 1}} \\ &= \frac{1}{\dfrac{1}{e^{\epsilon/k_\text{B}T} - 1}} = \exp\left(\frac{\epsilon}{k_\text{B}T}\right) - 1 \\ &= \exp(394.719) = 2.656 \times 10^{171}\,.\end{aligned} \tag{25}$$

2. Using the formula obtained at the previous point we generate the following equa-

tion:

$$
\begin{aligned}
\frac{A_{21}}{B_{21}u(\epsilon,T)} &= \exp\left(\frac{\epsilon}{k_{\mathrm{B}}T}\right) - 1 = 1\,, \\
\exp\left(\frac{\epsilon}{k_{\mathrm{B}}T}\right) &= 2\,, \\
\frac{\epsilon}{k_{\mathrm{B}}T} &= \ln 2\,, \\
k_{\mathrm{B}}T &= \frac{\epsilon}{\ln 2}\,, \\
T &= \frac{\epsilon}{k_{\mathrm{B}}\ln 2} = 170838\ \mathrm{K}\,.
\end{aligned}
$$

17 Populations Evolving in Time

Consider the $6s_{1/2} \to 6p_{1/2}$ excitation in Cs atomic vapor, with wavelength $\lambda = 890$ nm. Assume a natural lifetime $\tau = 2.2 \times 10^{-7}$ s for $6p_{1/2}$.
Determine:

1. the time it takes for the excited-state population to decrease from an initial 90% to a final 60%, assuming complete absence of incoming resonant radiation of wavelength λ;
2. how would this time change in the presence of equilibrium blackbody radiation generated by a cavity at temperature $T = 3500$ K.

17.1 Solution

Call the $6s_{1/2}$ state "1" and the $6p_{1/2}$ state "2".
The equations describing how the numbers of atoms in states 1 and 2 change in time are:

$$\frac{dN_1}{dt} = -B_{12}\rho(\epsilon)N_1 + (A_{21} + B_{21}\rho(\epsilon))N_2 \tag{26}$$

$$\frac{dN_2}{dt} = B_{12}\rho(\epsilon)N_1 - (A_{21} + B_{21}\rho(\epsilon))N_2\,. \tag{27}$$

Clearly,

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} = \frac{dN}{dt} = 0\,,$$

indicating that no Cs atom gets lost nor created in these transitions.
For this transition

$$\epsilon = \frac{hc}{\lambda} = 2.23196 \times 10^{-19}\ \mathrm{J} = 1.393\ \mathrm{eV}$$

and the ratio

$$\frac{\epsilon}{k_{\mathrm{B}}T} = \frac{2.23196 \times 10^{-19}\ \mathrm{J}}{4.832 \times 10^{-20}\ \mathrm{J}} = \frac{1.393\ \mathrm{eV}}{0.3016\ \mathrm{eV}} = 4.61887$$

will certainly turn out handy in the following.

1. For this part of the problem, the radiation intensity $\rho(\epsilon) = 0$. Accordingly, the rate equations simplify to:

$$
\begin{aligned}
\frac{dN_1}{dt} &= A_{21}N_2 = A_{21}(N - N_1) \\
\frac{dN_2}{dt} &= -A_{21}N_2\,.
\end{aligned}
$$

We focus on the second equation, whose solution is the standard exponential decay:

$$N_2(t) = N_2(0)e^{-A_{21}t}.$$

$\tau = A_{21}^{-1}$ is precisely the natural *lifetime* of this spontaneous decay phenomenon.
In this problem's conditions, $N_2(0) = 90\%\,N$.
The requested $t_{60\%}$ is determined by

$$60\%\,N = N_2(t_{60\%}) = N_2(0)e^{-A_{21}t_{60\%}} = 90\%\,N\,e^{-A_{21}t_{60\%}},$$

i.e.

$$\frac{60\%\,N}{90\%\,N} = e^{-A_{21}t_{60\%}},$$

or

$$\ln\frac{2}{3} = -A_{21}t_{60\%} = -\frac{t_{60\%}}{\tau}\,,$$

$$t_{60\%} = \tau\,\ln\frac{3}{2} = \tau \times 0.405465 = 8.9202 \times 10^{-8}\ \mathrm{s}^{-1}\,.$$

2. In the general case one needs the full equations (26) and (27).
Due to the conservation of the total number of Cs atoms, one can rewrite them as a single equation, e.g. the equation for the number of excited atoms:

$$\begin{aligned}\frac{dN_2}{dt} &= B_{12}\rho(\epsilon)(N - N_2) - (A_{21} + B_{21}\rho(\epsilon))N_2 \\ &= B\rho(\epsilon)(N - N_2) - (A_{21} + B\rho(\epsilon))N_2 \\ &= B\rho(\epsilon)N - (A_{21} + 2B\rho(\epsilon))N_2\,,\end{aligned} \tag{28}$$

where we have used the Einstein equation $B_{12} = B_{21} = B$.
This is a first-order equation with a stable fixed point (i.e. a constant solution, with $\frac{dN_2}{dt} = 0$) given by

$$N_2(t) \equiv \bar{N}_2 = \frac{B\rho(\epsilon)N}{A_{21} + 2B\rho(\epsilon)} = \frac{N}{2 + \frac{A_{21}}{B\rho(\epsilon)}} = \frac{N}{2 + \frac{A_{21}}{Bu(\epsilon,T)}}\,.$$

In the denominator of the last expression we have reconstructed the ratio of the spontaneous to stimulated emission, evaluated in a preceding problem to $\exp\left(\epsilon/(k_\mathrm{B}T)\right) - 1$. Substituting in the equation, we obtain:

$$\bar{N}_2 = \frac{N}{2 + \exp\left(\frac{\epsilon}{k_\mathrm{B}T}\right) - 1} = \frac{N}{\exp\left(\frac{\epsilon}{k_\mathrm{B}T}\right) + 1} = 0.0097676 \times N\,,$$

using this problem's data. Note that this expression is (of course!) consistent with the Boltzmann equilibrium distribution, since

$$\bar{N}_2 = \frac{N}{\exp\left(\frac{\epsilon}{k_\mathrm{B}T}\right) + 1} = N \times \frac{\exp\left(-\frac{\epsilon}{k_\mathrm{B}T}\right)}{1 + \exp\left(-\frac{\epsilon}{k_\mathrm{B}T}\right)} = N \times \frac{\exp\left(-\frac{\epsilon}{k_\mathrm{B}T}\right)}{Z}$$

Using the asymptotic value $\bar{N}_2$, the general solution to the differential equation (28) is conveniently written as

$$N_2(t) = \bar{N}_2 + (N_2(0) - \bar{N}_2)e^{-(A_{21}+2B\rho(\epsilon))t}\,.$$

Both the initial condition $N_2(0)$ and the asymptotic value $\bar{N}_2$ are obviously satisfied. To check that this expression solves the equation (28), it is sufficient to substitute it.

Now, for the problem's conditions, as above

$$\begin{aligned}60\%\, N = N_2(t_{60\%}) &= \bar{N}_2 + (N_2(0) - \bar{N}_2)e^{-(A_{21}+2B\rho(\epsilon))t_{60\%}} \\ &= \bar{N}_2 + (90\%\, N - \bar{N}_2)e^{-(A_{21}+2B\rho(\epsilon))t_{60\%}} .\end{aligned}$$

Whence,

$$\begin{aligned}60\%\, N - \bar{N}_2 &= (90\%\, N - \bar{N}_2)e^{-(A_{21}+2B\rho(\epsilon))t_{60\%}} \\ \frac{60\%\, N - \bar{N}_2}{90\%\, N - \bar{N}_2} &= e^{-(A_{21}+2B\rho(\epsilon))t_{60\%}} \\ -(A_{21}+2B\rho(\epsilon))t_{60\%} &= \ln \frac{60\%\, N - \bar{N}_2}{90\%\, N - \bar{N}_2} \\ (A_{21}+2B\rho(\epsilon))t_{60\%} &= \ln \frac{90\%\, N - \bar{N}_2}{60\%\, N - \bar{N}_2} \\ t_{60\%} &= \frac{1}{A_{21}+2B\rho(\epsilon)} \ln \frac{90\% - \frac{\bar{N}_2}{N}}{60\% - \frac{\bar{N}_2}{N}} .\end{aligned}$$

We can simplify this expression by evaluating the denominator

$$\begin{aligned}A_{21}+2B\rho(\epsilon) = A_{21}+2Bu(\epsilon,T) &= A_{21}\left(1+2\frac{Bu(\epsilon,T)}{A_{21}}\right) \\ &= \tau^{-1}\left(1+2\frac{1}{e^{\frac{\epsilon}{k_B T}}-1}\right) = \tau^{-1}\left(\frac{e^{\frac{\epsilon}{k_B T}}-1+2}{e^{\frac{\epsilon}{k_B T}}-1}\right) \\ &= \tau^{-1}\frac{e^{\frac{\epsilon}{k_B T}}+1}{e^{\frac{\epsilon}{k_B T}}-1} = \tau^{-1}\frac{e^{\frac{\epsilon}{2k_B T}}+e^{-\frac{\epsilon}{2k_B T}}}{e^{\frac{\epsilon}{2k_B T}}-e^{-\frac{\epsilon}{2k_B T}}} \\ &= \frac{1}{\tau \tanh\left(\frac{\epsilon}{2k_B T}\right)} .\end{aligned}$$

With this problem's data:

$$\tanh\left(\frac{\epsilon}{2k_B T}\right) = \tanh(2.30943) = 0.98046 .$$

In conclusion,

$$\begin{aligned}t_{60\%} &= \tau \tanh\left(\frac{\epsilon}{2k_B T}\right) \ln \frac{90\% - \frac{1}{\exp\left(\frac{\epsilon}{k_B T}\right)+1}}{60\% - \frac{1}{\exp\left(\frac{\epsilon}{k_B T}\right)+1}} \\ &= 2.2\times 10^{-7}\ \mathrm{s} \times 0.98046 \times \ln\frac{0.9-0.0097676}{0.6-0.0097676} \\ &= 2.1570\times 10^{-7}\ \mathrm{s}\times \ln(1.50827) \\ &= 8.866\times 10^{-8}\ \mathrm{s} ,\end{aligned}$$

in the same ballpark as τ, as expected.

18 Spontaneous and Stimulated Emission

Assume that of the total solar mass $M_{\text{Sun}} \simeq 2 \times 10^{30}$ kg, approximately 10% is atomic H in states 1s or 2p.

Assume also that this atomic gas is at equilibrium with thermal radiation at $T_{\text{Sun}} \simeq$ 6000 K.

How many H atoms per second are spontaneously decaying 2p→1s and how many are decaying due to stimulated emission?

18.1 Solution

$$N = \frac{2 \times 10^{30}\ \text{kg}}{1\ \text{a.m.u.}} \times 10\% = 1.204 \times 10^{56}$$

H atoms in total, distributed between 1s and the 3 2p states.

As in the previous problem, the excitation energy is

$$\epsilon = \frac{3}{8} E_{\text{Ha}} = 1.6349 \times 10^{-18}\ \text{J} = 10.204\ \text{eV}$$

(nonrelativistic approximation and reduced mass $\mu \simeq m_e$), so that the ratio

$$\frac{\epsilon}{k_{\text{B}} T} = 19.7359\,.$$

Call N_1 = number of atoms in 1s and N_2 = number of atoms in 2p. According to Boltzmann statistics,

$$N_1 = N \frac{1}{1 + 3\exp(-\epsilon/(k_{\text{B}}T))} = 1.204 \times 10^{56}$$
$$N_2 = N \frac{3\exp(-\epsilon/(k_{\text{B}}T))}{1 + 3\exp(-\epsilon/(k_{\text{B}}T))} = 9.698 \times 10^{47}$$

because $\exp(-\epsilon/(k_{\text{B}}T)) = 2.68403 \times 10^{-9}$, and taking into account the degeneracy 3 of level 2p.

A couple of months back, the single-atom 2p→1s rate of spontaneous decay has been calculated for a problem relative to 1-electron atoms:

$$A_{21} = \gamma_{\text{2p}\to\text{1s}} = 6.26832 \times 10^{8}\ \text{s}^{-1}\,.$$

Accordingly, the total rate of spontaneous decay is

$$R_{2\to 1}^{\text{tot spont}} = N_2\, A_{21} = 6.079 \times 10^{56}\ \text{s}^{-1}.$$

To evaluate the rate of emission stimulated by the equilibrium radiation, we take advantage of the formula for the ratio of the spontaneous to stimulated emission obtained in the previous problem.

$$\begin{aligned}
R_{2\to 1}^{\text{tot stimul}} &= N_2\, B_{21} u(\epsilon, T) = N_2\, A_{21} \frac{B_{21}\, u(\epsilon, T)}{A_{21}} \\
&= N_2\, A_{21} \frac{1}{e^{\epsilon/k_{\text{B}}T} - 1} \\
&= 6.079 \times 10^{56}\ \text{s}^{-1} \times 2.68403 \times 10^{-9} = 1.632 \times 10^{48}\ \text{s}^{-1}.
\end{aligned}$$

19 Statistics and Dynamics of Nuclear Magnetic Resonance (NMR)

An organic sample contains 5.0×10^{28} protons/m^3. Assume that the corresponding magnetic moments do not interact and reach equilibrium independently in a field $B =$ 1 T at room temperature $T = 300$ K. The magnetic moment of the proton is $\mu_z = \frac{1}{2} g\mu_B = 1.521 \times 10^{-3}\, \mu_B$.

1. Evaluate the absolute equilibrium magnetization in A/m, and
2. Express this magnetization as a fraction of the full-saturation magnetization expected at $T = 0$ K.
3. What is the resonant radiation frequency for the spin-flip transition?
4. For this transition, evaluate the ratio between spontaneous and stimulated emission at equilibrium.
5. Assume then that resonant electromagnetic radiation is sent on the material with an energy density equal to 10 times the value appropriate for equilibrium at room temperature: how does the steady-state magnetization change?

19.1 Solution

Energy levels:

$$\mathcal{E}_{\pm 1/2} = \pm\frac{1}{2} g\mu_B B = \pm 1.521 \times 10^{-3} \mu_B B = \pm 1.4106 \times 10^{-26}\ \text{J} = \pm 88.04\ \text{neV}\,.$$

Transition energy:

$$\Delta E = \mathcal{E}_{1/2} - \mathcal{E}_{-1/2} = g\mu_B B = 2.8212 \times 10^{-26}\ \text{J} = 176.08\ \text{neV}\,.$$

Dimensionless ratio:

$$x = \frac{\Delta E}{k_B T} = \frac{g\mu_B B}{k_B T} = \frac{2.8212 \times 10^{-26}\ \text{J}}{4.14195 \times 10^{-21}\ \text{J}} = 6.81118 \times 10^{-6}.$$

Partition function:

$$Z_{1\,\text{spin}} = \sum_{M_J=-J}^{J} \exp\left(-x\, M_J\right), \tag{29}$$

that for this spin-$1/2$ problem is:

$$Z_{1\,\text{spin}} = \exp\left(-x\,(-1/2)\right) + \exp\left(-x\, 1/2\right) = e^{x/2} + e^{-x/2} = 2\cosh(x/2) \tag{30}$$

Equilibrium magnetization derived in an early problem:

$$M_z^{\text{eq}} = \frac{N}{V}\left[\mu_{1\,z}\right] = -\frac{N}{VB} U_{1\,\text{spin}} = \frac{g\mu_B N}{2V} \tanh\frac{x}{2}\,. \tag{31}$$

1. In the calculation of the magnetization, at room temperature $x \ll 1$, therefore one can expand the hyperbolic tangent:

$$\begin{aligned} M_z^{\text{eq}} &= \frac{g\mu_B N}{2V} \tanh\frac{x}{2} \simeq \frac{g\mu_B N}{2V}\frac{x}{2} \\ &= 705.2\ \frac{\text{A}}{\text{m}} \times 3.40559 \times 10^{-6} = 0.0024019\ \frac{\text{A}}{\text{m}}\,. \end{aligned} \tag{32}$$

2. The saturation magnetization is obtained for $x \to \infty$, which implies $\tanh x/2 \to 1$. Therefore the saturation magnetization is

$$M_z^{\text{satur}} = \frac{g\mu_{\text{B}}N}{2V} = 705.2\ \frac{\text{A}}{\text{m}}\,.$$

Accordingly,

$$\frac{M_z^{\text{eq}}}{M_z^{\text{satur}}} = \frac{x}{2} = 3.40559 \times 10^{-6}.$$

3.

$$\nu = \frac{\Delta E}{2\pi\hbar} = 42.5766\ \text{MHz}\,.$$

4. According to the calculations of Eq. (25), at equilibrium

$$\begin{aligned}\frac{A_{1/2\to-1/2}}{B_{1/2\to-1/2}u(\Delta E,T)} &= e^x - 1 \\ &\simeq 1 + x - 1 = x = 6.81118 \times 10^{-6}\,.\end{aligned}$$

This indicates that stimulating emission is overwelmingly dominating over spontaneous emission, as is usually the case for low-energy spectroscopies. This also the opposite of what we found for the 2p→2s transitions in the previous problem.

5. Rate equations:

$$\frac{dN_{-1/2}}{dt} = -B_{-1/2\to1/2}\rho(\Delta E)N_{-1/2} + (A_{1/2\to-1/2} + B_{1/2\to-1/2}\rho(\Delta E))N_{1/2} \tag{33}$$
$$\frac{dN_{1/2}}{dt} = B_{-1/2\to1/2}\rho(\Delta E)N_{-1/2} - (A_{1/2\to-1/2} + B_{1/2\to-1/2}\rho(\Delta E))N_{1/2}\,. \tag{34}$$

The steady state corresponds to

$$\frac{dN_{M_J}}{dt} = 0\,.$$

From the first equation,

$$-B_{-1/2\to1/2}\rho(\Delta E)N_{-1/2} + (A_{1/2\to-1/2} + B_{1/2\to-1/2}\rho(\Delta E))N_{1/2} = 0\,.$$

$$B_{-1/2\to1/2}\rho(\Delta E)N_{-1/2} = (A_{1/2\to-1/2} + B_{1/2\to-1/2}\rho(\Delta E))N_{1/2}\,.$$

$$\frac{N_{1/2}}{N_{-1/2}} = \frac{B_{-1/2\to1/2}\rho(\Delta E)}{A_{1/2\to-1/2} + B_{1/2\to-1/2}\rho(\Delta E)}\,.$$

According to Einstein's equations, the two B's are the same:

$$\frac{N_{1/2}}{N_{-1/2}} = \frac{B\rho(\Delta E)}{A_{1/2\to-1/2} + B\rho(\Delta E)}\,.$$

$$\frac{N_{1/2}}{N_{-1/2}} = \frac{1}{\dfrac{A_{1/2\to-1/2}}{B\rho(\Delta E)} + 1}\,. \tag{35}$$

A quantity similar to the fraction at the denominator has been computed at the previous point, with $u(\Delta E, T)$ in place of $\rho(\Delta E)$.

That ratio was very small already and, because $\rho(\Delta E) = 11\, u(\Delta E, T) > u(\Delta E, T)$,[1] the ratio at the denominator of Eq. (35) is even smaller. Therefore we can certainly use the standard small-ϵ expansion $(1+\epsilon)^{-1} \simeq 1 - \epsilon + O(\epsilon^2)$:

$$\frac{N_{1/2}}{N_{-1/2}} = \frac{1}{1 + \dfrac{A_{1/2\to-1/2}}{B\rho(\Delta E)}} \simeq 1 - \frac{A_{1/2\to-1/2}}{B\rho(\Delta E)} = 1 - \frac{A_{1/2\to-1/2}}{B \times 11\, u(\Delta E, T)}$$
$$= 1 - \frac{1}{11}\,\frac{A_{1/2\to-1/2}}{B\, u(\Delta E, T)} \simeq 1 - \frac{1}{11}\, x\,.$$

The steady-state magnetization is therefore:

$$\begin{aligned} M_z &= -\frac{g\mu_{\mathrm{B}}}{2V}\,\left(N_{1/2} - N_{-1/2}\right) = -\frac{g\mu_{\mathrm{B}}\, N_{-1/2}}{2V}\left(\frac{N_{1/2}}{N_{-1/2}} - 1\right) \\ &\simeq -\frac{g\mu_{\mathrm{B}}\, N/2}{2V}\left(1 - \frac{1}{11}\, x - 1\right) = -\frac{g\mu_{\mathrm{B}}\, N/2}{2V} \times \left(-\frac{1}{11}\, x\right) \\ &= \frac{g\mu_{\mathrm{B}}\, N/2}{2V} \times \left(\frac{1}{11}\, x\right) = \frac{g\mu_{\mathrm{B}}\, N}{2V}\,\frac{x}{2} \times \frac{1}{11} = M_z^{\mathrm{eq}} \times \frac{1}{11} \\ &= 0.0024019\,\frac{\mathrm{A}}{\mathrm{m}} \times \frac{1}{11} = 0.00021836\,\frac{\mathrm{A}}{\mathrm{m}}\,. \end{aligned} \tag{36}$$

We see that the resonating radiation has the effect of inducing a significant *demagnetization* compared to the equilibrium state.

[1] This factor 11 comes from the external irradiation at a power level $10\, u(\Delta E, T)$ plus the regular thermal irradiation at a power level $u(\Delta E, T)$.